Analytic Geometry and Calculus I

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Analytic Geometry and Calculus I: Workbook

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The courses align to career pathways in New Jersey's growth industries including health services, technology, energy, and global manufacturing and supply chain management as identified by the New Jersey Council of Community Colleges.

Introduction

In this workbook, the team* provides supplement material to the OpenStax Calculus** textbook, and our aim is for it to be used in a standard Calculus 1 course or equivalent. In each section, we provide students definitions or theorems along with examples and practice problems. In chapter 1, we cover limits and continuities. In chapter 2, we focus mainly on derivatives and differentiation techniques. In chapter 3, we explore applications of differentiation and in chapter 4, we tackle indefinite and definite integrals, as well as integration by substitution.

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CHAPTER 1: LIMITS AND CONTINUITY

Every house needs a solid foundation and limits are the foundation on which the “house” of calculus is built. In this chapter, you will investigate the limits of function and you will also learn what it means for a function to be continuous.

Learning Objectives:

- Understand the intuition behind the idea of “limit” by estimating the instantaneous velocity of an object.
- Determine the existence of a limit of a function.
- Determine limits of functions using graphic and numeric methods.
- Determine the limits of functions analytically.
- Determine the limits of functions that initially present as indeterminate forms.
- Test a function for continuity at a point and continuity on an interval.
- Evaluate limits of continuous functions.
- Use variable substitution to evaluate special limits for trigonometric functions.

Introduction to Calculus and Finding Limits with Tables of Values

Key Questions in Calculus

- How do we determine the slope of a line tangent to a curve at one point?
- How do we determine the area under a curve?

The slope of the tangent line to a curve can be found by calculating the ***derivative*** of the curve, which can be evaluated at the x-value where the tangent line touches it. This will be discussed later. However, the main idea is that we can use Calculus to find the rate of change of moving objects (i.e. their instantaneous velocities and average velocities).

In order to do so, we must first gain an intuitive and practical understanding of limits.

Definition: Let f(x) be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. If *all* values of the function f(x) approach the real number L as the values of x ($\neq$a) approach the number a, then we say that the limit of f(x) as x approaches a is L. (More succinctly, as x gets closer to a, f(x) gets closer and stays close to L.) Symbolically, we express this idea as $\lim_{x \to a} f(x) = L$.

One way to determine a limit is to plug in x-values that continue to get closer and closer to the a-value that x is approaching. In many cases, f(x) will also approach a finite number, which is the limit of the function at a.

Try It: Fill in the table to determine the limit of f(x) = $\frac{x^2 - 4}{x - 2}$ as x approaches 2.

x	1.99	1.999	1.9999	2	2.0001	2.001	2.01
f(x)							

Note that f(x) is undefined at x = 2 (with a removable discontinuity), but the limit still exists at 2.

f(x) does not have to be defined at the limit value for the limit to exist!

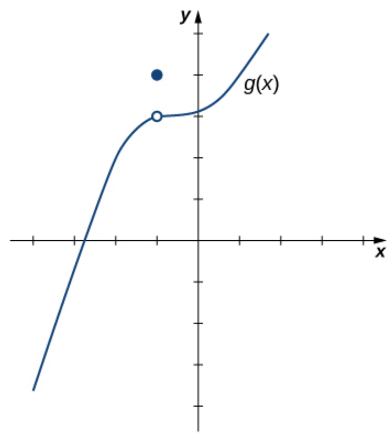
Try It: Fill in the table to determine the limit of f(x) = $\frac{\sin(x)}{x}$ as x approaches 0.

x	-0.01	-0.001	-0.0001	0	.0001	0.001	0.01
f(x)							

Finding Limits from Graphs, Sided Limits, and Existence of a Limit

Of course, there are other ways to determine limits, many of which are much less time-consuming and pedantic than using a table of values. One way to do so is graphically. Often, by simply looking at a graph, we can determine the limit of a function as x approaches a certain value.

Try It: Use the graph of g(x) below to determine the limit of g(x) as x approaches -1. $\lim_{x \to -1} g(x) =$



Sided Limits: So far, we have looked at limits being approached from two sides (namely, two-sided limits). However, we can also evaluate one-sided limits. These will be denoted by a superscript + sign on the number (meaning from the right) or a superscript – sign (meaning from the left).

Try It: Use a table of values to evaluate $f(x) = \frac{|x^2 - 4|}{x - 2}$ as x approaches 2 from the left.

Try It: Use a table of values to evaluate $f(x) = \frac{|x^2 - 4|}{x - 2}$ as x approaches 2 from the right.

Note that you should not have gotten the same value for L, meaning the two-sided limit as x approaches 2 does not exist. In order for it to exist, the limit from the left must equal the limit from the right.

Existence of a Limit: There are certain instances, as we just saw, where the limit does not exist.

The limit of a function does not exist if…

- The limit from the left does not equal the limit from the right (i.e., a jump discontinuity).
- The graph "blows up" to ∞ or - ∞ at a vertical asymptote.
- The graph has some oscillatory behavior near a point.
- The graph does not exist in an open interval around a point in its domain (i.e. $\sqrt{x}$ does not exist in any open interval containing 0).

Try It: For the following limits, determine whether or not the limit exists. If it does, find it.

If not, show why and write "DNE".

a. $\lim_{x \to 0} \frac{1}{x} =$

b. $\lim_{x \to 0} \frac{\sin x}{x} =$

c. $\lim_{x \to 0} \sqrt{x} =$

Laws of Limits and Limit Evaluation

Lets say you had a thin piece of paper, so thin in fact it is only a molecule thick. You stand on the far right side of the room, and decide you will walk half the distance to the left side of the room. You then decide to walk half the distance from your current position to the left side. You keep doing this operation to bring the paper to the left wall and ask yourself "Will the paper ever actually reach the other wall?" Based on what you did, it will never actually make it to the other wall, because each time you only walk half the distance to the left wall, so no matter how many times you do this you still have a half of some amount to go. You can, however, get as close to the wall as you like by just iterating this process. That is what a limit is: **A limit is where you are going, not where your destination is**. This informal idea helps us think about limits outside of the definition, and leads us to our limit laws.

Let $f(x)$ and $g(x)$ be defined on an open interval I for every value of x, except possibly at $x = a$. Let c be any real number constant. If

$$\lim_{x\to a} f(x) = L \text{ and } \lim_{x\to a} g(x) = M$$

then

1. $\lim_{x\to a} c = c$

   The limit of a constant is the constant

2. $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x) = cL$

   Having a coefficient with our function just multiples the limit of the function by the coefficient

3. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$

   Limit of the sum/difference is the sum/difference of the limits

4. $\lim_{x\to a}[f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = L \cdot M$

   The limit of the product is the product of the limits

5. $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}, M \neq 0$

   The limit of the quotient is the quotient of the limits

6. $\lim_{x\to a} f^n(x) = \left(\lim_{x\to a} f(x)\right)^n = L^n$, for any positive integer n

   The limit of a power is the power of the limit

7. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$, provided the root exists

   Limit of a root is the root of the limit

It can be helpful to think of the rules in terms of what they are doing. Some rules tell us that our normal binary operations do not change the outcome of limits. Contants, coefficients, and powers to not affect the method of solving, just how much of the solution we get. What you have together, you can solve separately. This means to evaluate a limit, we can do it piece by piece incrementally. Though we will write it out for our examples, in practice you want to be comfortable using this without having to rewrite everything

Examples

1. $\lim_{x\to 3}(x^2+2x)$

$$= \lim_{x\to 3} x^2 + \lim_{x\to 3} 2x$$

$$= (\lim_{x\to 3} x)^2 + 2\lim_{x\to 3} x$$

$$= 3^2 + 2(3) = 15$$

2. $\lim_{x\to -1}[(x-5)^3(4x^2+2x-3)]$

$$= \lim_{x\to -1}(x-5)^3 \cdot \lim_{x\to -1}(4x^2+2x-3)$$

$$= (-1-5)^3 \cdot (4(-1)^2+2(-1)-3) = -216 \cdot -1 = 216$$

3. $\lim_{x\to 9}\sqrt{\dfrac{(x+6)}{(x-3)}}$

$$= \sqrt{\lim_{x\to 9}\left[\frac{(x+6)}{(x-3)}\right]}$$

$$= \sqrt{\frac{\lim_{x\to 9}(x+6)}{\lim_{x\to 9}(x-3)}}$$

$$= \sqrt{\frac{9+6}{9-3}} = \sqrt{\frac{15}{6}} = \sqrt{\frac{5}{2}}$$

Notice we are essentially plugging the value of the limit in for x, the same way you would evaluate $f(x)$ at a value.

Practice

For the following functions, calculate the limit:

1. $\lim_{x \to 5} 7x^2 - 8x + 3$

2. $\lim_{x \to -2} [(x+4)^2(-x^2 + 3x + 1)]$

3. $\lim_{t \to 3} \frac{\sqrt{x+6}}{x-5}$

4. $\lim_{x \to \frac{\pi}{3}} 2\sin(x)\cos(x)$

Indeterminant Forms and Squeeze Theorem

When taking a limit, we know we attempt to evaluate the function at the value the limit is approaching. There is always a chance this does not give a determined solution. This leads to the indeterminant forms

The indeterminant forms are $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty, 0 \cdot \infty,\ 0^0,\ 1^\infty,\ \infty^0$

We will consider only the first three forms, as the others use methods we will not discuss here. These forms are indeterminant because depending on the limit being taken, the answer from these forms will change.

To solve an indeterminant form $\frac{0}{0}$

1. If the problem is a rational function, we factor a root term out of the numerator and denominator to simplify and solve the limit
2. With square roots, multiple by the conjugate to alter the problem

Examples

1. Find the limit:

$$\lim_{x \to 5} \frac{x^2 - 25}{x - 5}$$

Solution:
If we evaluate the limit we see we get $\frac{0}{0}$, which is an indeterminant form. We can factor the numerator

$$\lim_{x \to 5} \frac{(x-5)(x+5)}{x-5}$$

Since they have the same factor in common, we can cancel it

$$\lim_{x \to 5} \frac{\cancel{(x-5)}(x+5)}{\cancel{x-5}} = \lim_{x \to 5}(x+5) = 10$$

If the numerator and denominator were both made zero when 5 was substituted in, then 5 must be a root of the polynomial. This means when factored, we will have a common factor to cancel that was making our problem 0, in this case $x - 5$.

2. Find the limit:

$$\lim_{x\to 0} \frac{\sqrt{x^2+25}-5}{x}$$

Solution:
If we evaluate the limit we see we get $\frac{0}{0}$, which is indeterminant, but factoring will not work. In this case, consider what is causing the issue. The numerator being a difference of two terms make it zero, and the square root prevents us from manipulating it. To work with this one, we will multiply by the conjugate of the numerator

$$\lim_{x\to 0} \frac{\sqrt{x^2+25}-5}{x} \cdot \frac{\sqrt{x^2+25}+5}{\sqrt{x^2+25}+5}$$

Remember that $(a+b)(a-b) = a^2 - b^2$ and $(\sqrt{x})^2 = x$ for positive x values

$$\lim_{x\to 0} \frac{x^2+25-25}{x(\sqrt{x^2+25}+5)} = \lim_{x\to 0} \frac{x^2}{x(\sqrt{x^2+25}+5)} = \lim_{x\to 0} \frac{x}{(\sqrt{x^2+25}+5)} = \frac{0}{10} = 0$$

The numerator and denominator were still zero, but factoring was not an option. Since there was a square root, we can use a conjugate to simplify the problem

The indeterminant form $\frac{\infty}{\infty}$ comes most from limits at infinity. A useful strategy is:

For a rational function:

$$\lim_{x\to\infty} \frac{ax^n+\cdots}{bx^n+\cdots}$$

1. If $n > m$ then the limit goes to infinity
2. If $n < m$ then the limit goes to zero
3. If $n = m$ then the limit goes to $\frac{a}{b}$

This idea will hold for limits going to $-\infty$, though we need to carefully consider the signs from the original problem to determine if the result is positive or negative.

Examples

1. Find the limit:

$$\lim_{x\to\infty} \frac{x^2+3x-4}{5x^3-x^2+8}$$

   Solution:
   If we evaluate the limit we see we get $\frac{\infty}{\infty}$, which is an indeterminant form. Notice the powers of the numerator and denominator

$$2 < 3$$

   Since the denominator is a higher power than the numerator, the limit goes to 0

$$\lim_{x\to\infty} \frac{x^2+3x-4}{5x^3-x^2+8} = 0$$

2. Find the limit:

$$\lim_{x\to 0} \frac{\sqrt{3x^2+25}}{x+2}$$

   Solution:
   If we evaluate the limit we see we get $\frac{\infty}{\infty}$, which is an indeterminant form. This may not be a rational function, when a polynomial goes to $\pm\infty$, only the highest power term matters. That means for the problem

$$\lim_{x\to\infty} \frac{\sqrt{3x^2+25}}{x+2} \text{ behaves like } \lim_{x\to\infty} \frac{\sqrt{3x^2}}{x}$$

   so the numerator is really to the first degree, and the denominator is to the first degree. Since they are the same, it goes to leading coefficients.

$$\lim_{x\to\infty} \frac{\sqrt{3x^2+25}}{x+2} = \sqrt{3}$$

   The method of compairing powers works for rational functions and those that behave like rationals as we move towards $\pm\infty$

To solve an indeterminant form $\infty - \infty$

1. If the problem has fractions, combine the fractions to see if anything can cancel
2. If there is no fraction, multiply the problem by the conjugate over itself

Examples

1. Find the limit:

$$\lim_{x\to\infty} \sqrt{x^2+3x} - x$$

Solution:
If we evaluate the limit we see we get $\infty - \infty$, which is an indeterminant form. The problem occurs because of the minus between the terms. If we multiply by the conjugate, we can simplify our problem

$$\lim_{x\to\infty} \frac{\sqrt{x^2+3x} - x}{1} \cdot \frac{\sqrt{x^2+3x} + x}{\sqrt{x^2+3x} + x}$$

The denominator will work well, as it is $\infty + \infty$. The numerator will simply same as we used before

$$\lim_{x\to\infty} \frac{x^2+3x-x^2}{\sqrt{x^2+3x}+x} = \lim_{x\to\infty} \frac{3x}{\sqrt{x^2+3x}+x}$$

Recall from our previous section, if the degree in the numerator is equal to the degree in the denominator, the limit as $x \to \infty$ will go to leading coefficients. The degree in both the numerator and denominator is 1, since

$$\lim_{x\to\infty} \sqrt{x^2+3x} + x = \lim_{x\to\infty} \sqrt{x^2} + x = \lim_{x\to\infty} 2x$$

So the degree of the numerator and denominator are equal, it goes to leading coefficients

$$\lim_{x\to\infty} \sqrt{x^2+3x} - x = \lim_{x\to\infty} \frac{3x}{\sqrt{x^2+3x}+x} = \frac{3}{2}$$

Point: When it comes to squeeze theorem, we want to to bind the unknown limit. This usually requires with starting with a simpler portion of it, and building up through binary operations to the limit we want. Often, this is used to prove trigonometric limits, since trigonometric functions are cyclic in nature.

Practice

For the following functions, calculate the limit:

1. $\lim_{x \to 6} \dfrac{x^2 - 7x + 6}{x^2 - 2x - 24}$

2. $\lim_{x \to \infty} \dfrac{(x+4)^2}{3x^2 + x - 1}$

3. $\lim_{x \to \infty} \sqrt{x^2 + 4} - x$

4. $\lim_{x \to \frac{\pi}{2}} \dfrac{\sin(2x)}{\cos(x)}$

Infinite Limits and Limits at Infinity

Infinite Limits: Limits do not have to be equal to finite numbers. They can be equal to ∞ or - ∞.

We say that $\lim_{x\to a} f(x) = \infty$ if we can make f(x) arbitrarily large as we approach a from both sides without actually letting x = a.

Similarly, the limit can equal - ∞ if we can make f(x) arbitrarily large and negative, and follow the same definition as above.

Try It: Evaluate $\lim_{x\to 0^+} \frac{6}{x^2}$, $\lim_{x\to 0^-} \frac{6}{x^2}$, and $\lim_{x\to 0} \frac{6}{x^2}$

Try It: Evaluate $\lim_{x\to -2^+} \frac{-4}{x+2}$, $\lim_{x\to -2^-} \frac{-4}{x+2}$, and $\lim_{x\to -2} \frac{-4}{x+2}$

Properties:

Suppose that we have $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = L$ for some real numbers c and L. Then,

- $\lim_{x\to c} |f(x) \pm g(x)| = \infty$
- If L > 0, then $\lim_{x\to c} |f(x) * g(x)| = \infty$
- If L < 0, then $\lim_{x\to c} |f(x) * g(x)| = -\infty$
- $\lim_{x\to c} g(x)/f(x) = 0$
- $\lim_{x\to c} f(x)/g(x) = \infty$

Limits at Infinity: Infinite limits and limits at infinity are not one and the same. An infinite limit is a limit that is equal to infinity as x approaches some *c*. Whereas, a limit at infinity is one where x is actually approaching either ∞ or - ∞, and the limit value (which can be finite or infinite) is determined from there.

Properties:

If we have any limits of the form: $\lim_{x \to \infty} c/x^r$ or $\lim_{x \to -\infty} c/x^r$, these are equal to 0 because the denominator is getting arbitrarily large while the numerator is being held constant.

If we have anything of the form ∞ + ∞ or ∞ - ∞, note that these are both equal to ∞.

Try It: Evaluate $\lim_{x \to \infty} 2x^3 - x^2 + 8x =$

Note: If you have a limit at infinity of a fraction, then there are some shortcuts to take.

- If the highest exponent of the numerator is greater than the highest exponent of the denominator, then the limit at infinity is equal to infinity.
- If the highest exponent of the numerator is less than the highest exponent of the denominator, then the limit at infinity is equal to 0.
- If the highest exponent of the numerator is equal to the highest exponent of the denominator, then the limit at infinity is equal to the quotient of the leading coefficients.

Try It: Evaluate the following limits.

$$\lim_{x \to \infty} \frac{2x^4 - x^2 + 3x}{-7x^4 - 5x^3 + x} =$$

$$\lim_{x \to \infty} \frac{5x^{14} - 6x^2}{-9x^{34} - x^3} =$$

If we have limits at infinity involving *e*, then we need to evaluate the limit of the exponent at infinity. If that limit is +∞, then the limit is +∞. If it is -∞, then the limit is 0.

Try It: Evaluate the following limits.

$$\lim_{x\to\infty} e^{x^4 - x + 9} =$$

$$\lim_{x\to\infty} e^{9-x-x^4} =$$

Note: For the six basic trigonometric functions, the limits at infinity do not exist. However, the limit at infinity does exist for arctan. For +∞, the limit is $\frac{\pi}{2}$ and for - ∞, the limit is $-\frac{\pi}{2}$.

Continuity and Limits of Continuous Functions

Summary: Continuity of $f(x)$ at $x = a$
➢ Determine the value of $f(a)$. ➢ Determine $\lim_{x \to a} f(x)$. ➢ Check to see that both results are equal. If so, $f(x)$ is continuous at $x = a$.

Try It: Determine whether the function is continuous at the given value.	
$f(x) = \frac{3x^2 + 2x + 1}{x - 1}, x = 1$	$f(x) = \begin{cases} x \sin \frac{1}{x}, x \neq 0 \\ 0, x = 0 \end{cases}, x = 0$

(TO INSTRUCTIONAL DESIGN: CREATE SOLUTIONS LINK HERE)

Summary: Continuity on an Interval $[a, b]$
➢ Check to see if the function is right-continuous at $x = a$. ➢ Check to see if the function is left-continuous at $x = b$. ➢ Check to see if the function is continuous for any $x = c$ such that $a < c < b$. ○ Note: This procedure can be modified for any type of interval.

Try It: Determine whether the function is continuous on its domain.	
$f(x) = \sqrt{9 - x^2}$	$f(x) = (x^2 + x - 1)^{\frac{2}{3}}$

(TO INSTRUCTIONAL DESIGN: CREATE SOLUTIONS LINK HERE)

Summary: Removable vs. Non-Removable Discontinuities
➢ A removable discontinuity is one where $f(a)$ is undefined, but $\lim_{x\to a} f(x)$ exists. ➢ A non-removable discontinuity is one where $f(a)$ may exist, but $\lim_{x\to a} f(x)$ does not exist.

Try It: Classify the following points of discontinuity for $f(x) = \frac{x-3}{x^2-7x+12}$	
$x = 3$	$x = 4$

(TO INSTRUCTIONAL DESIGN: CREATE SOLUTIONS LINK HERE)

Summary: Limits of Continuous Functions
➢ Suppose that $y = f\big(g(x)\big)$. ○ If $g(x)$ is continuous at $x = a$ and $f(x)$ is continuous at $x = g(a)$, then $\lim_{x\to a} f\big(g(x)\big) = f\big(g(a)\big)$. ○ If $g(x)$ has a removable discontinuity at $x = a$ and if $f(x)$ is continuous at where $x = \lim_{x\to a} g(x)$, then $\lim_{x\to a} f\big(g(x)\big) = f\left(\lim_{x\to a} g(x)\right)$.

Try It: Find each limit	
$\lim_{x\to\frac{\pi}{2}} \frac{\cos^2 x - 1}{\cos x - 1}$	$\lim_{x\to -1} \sqrt{2x^2 - 1}$

(TO INSTRUCTIONAL DESIGN: CREATE SOLUTIONS LINK HERE)

SOLUTIONS

1. Try It for Continuity at a Point
   a. The function is not continuous at $x = 1$.
   b. The function is continuous at $x = 0$.

2. Try It for Continuity on an Interval.
   a. The function is continuous on its domain of $[-3,3]$.
   b. The function is continuous on its domain of $(-\infty, \infty)$.

3. Try It for Classifying Discontinuity
   a. Removable.
   b. Non-Removable.

4. Try It for Limits of Continuous Functions
   a. The limit is 1.
   b. The limit is 1.

Special Limits for Trigonometric Functions

Recall:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad\qquad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

Suppose the goal was to solve a limit problem such as these:

$$\lim_{x \to 0} \frac{\sin(2x)}{x}$$

$$\lim_{x \to 0} \frac{1 - \cos(4x)}{x}$$

We can obtain the result by using a **variable substitution**:

1. Take the argument of the trigonometric function and set it equal to a new variable. For instance, we can choose the variable u.

2. Use your substitution and solve for x. This new expression in terms of u will replace x in the denominator.

3. Take the limit of your substitution as x approaches zero. This will be the value to approach for your limit in terms of u. Rewrite the limit in terms of u.

4. Use limit laws and the special trigonometric limits to evaluate the new limit.

Guided Example: Evaluate the following limit.

$$\lim_{x \to 0} \frac{\sin(2x)}{x}$$

Step	**Work**
Take the argument of the trigonometric function and set it equal to a new variable.	$u = 2x$
Use your substitution and solve for x.	$x = \frac{u}{2}$
Take the limit of the substitution as x approaches zero and rewrite the limit.	$\lim_{x \to 0} 2x = 0$ $\lim_{x \to 0} \frac{\sin(2x)}{x} = \lim_{u \to 0} \frac{\sin u}{u/2}$
Evaluate the limit.	$\lim_{u \to 0} \frac{\sin u}{u/2} = 2 \lim_{u \to 0} \frac{\sin u}{u} = 2 \cdot 1 = 2$

Try It: Evaluate each limit. Be sure to show all the steps!		
$\lim_{x \to 0} \frac{\sin(5x)}{3x}$	$\lim_{x \to 0} \frac{1 - \cos(4x)}{x}$	$\lim_{x \to 0} \frac{\tan(2x)}{x}$

(INSTRUCTIONAL DESIGN: PUT SOLUTIONS LINK HERE)

Suppose we wished to calculate the following limit:

$$\lim_{x \to 0} \frac{\sin(4x)}{\sin(7x)}$$

In order to use the substitution procedure, we would need to introduce a denominator of x to numerator and denominator of the limit:

$$\lim_{x \to 0} \left(\frac{\sin(4x)}{\sin(7x)} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \right) = \lim_{x \to 0} \frac{\frac{\sin(4x)}{x}}{\frac{\sin(7x)}{x}}$$

Using the limit law for division, we can write this as two limits and perform the substitution procedure twice. Note that you should use different variables for your numerator substitution versus your denominator substitution.

$$\lim_{x \to 0} \frac{\frac{\sin(4x)}{x}}{\frac{\sin(7x)}{x}} = \frac{\lim_{x \to 0} \frac{\sin(4x)}{x}}{\lim_{x \to 0} \frac{\sin(7x)}{x}}$$

Complete the example in the space below!

Try It: Evaluate each limit. Be sure to show all the steps!		
$\lim_{x \to 0} \frac{\sin(8x)}{\sin(5x)}$	$\lim_{x \to 0} \frac{1 - \cos(3x)}{1 - \cos(9x)}$	$\lim_{x \to 0} \frac{\tan(6x)}{\tan(10x)}$

(INSTRUCTIONAL DESIGN: PUT SOLUTIONS LINK HERE)

Solutions to Try It Problems

1. First Try It Problem
    a. $\frac{5}{3}$
    b. 0
    c. 2

2. Second Try It Problem
    a. $\frac{8}{5}$
    b. 0
    c. $\frac{3}{5}$

CHAPTER 2: THE DERIVATIVE

The idea of the derivative deals with the concept of instantaneous rate-of-change which was initially presented in Chapter 1. Throughout this chapter, you will learn how to describe the derivative both numerically and as a function, as well as learn the various rules for the process of differentiation.

Learning Objectives:

1. Describe the derivative at a point as an instantaneous rate-of-change and as the slope of a tangent line to a curve.
2. Find derivatives for functions of all varieties (algebraic, trigonometric, inverse trigonometric, exponential, and logarithmic).
3. Apply the laws of differentiation to find derivatives of sums, differences, products, quotients, and composite functions.
4. Find derivatives for implicitly defined functions using implicit differentiation.
5. Solve simple application problems in physics and business involving derivatives and rates of change.

The Instantaneous Rate of Change and the Derivative

Consider driving from your house down the highway. Over the course of two hours, you have covered 120 miles. This means, on average you drove 60 miles per hour. We know though you did not drive 60 miles an hour the whole time, as you have started from your driveway and dare not do 60 on your street. If you averaged 60mph, you must have hit that value at some moment on the speedometer though. Thats the difference between the average rate of change, the slope of the secant line, and the instantaneous rate of change, the slope of the tangent line.

For any function $f(x)$, the equation of the tangent line at a point $P(x_0, f(x_0))$ is

$$y - f(x_0) = m_{tan}(x - x_0)$$

where

$$m_{tan} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

provided the limit exists.

Point: Recall that the slope of the tangent line at a point is the derivative of the function at the point. The two are the same.

Example

1. Find the tangent line for $f(x) = x^2$ at $x_0 = 2$
   Solution:
   $m_{tan} = \lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$

   $$= \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}} = \lim_{x \to 2} x + 2 = 4$$

   So the tangent line at $(2, 4)$ is

   $$y - 4 = 4(x - 2)$$

Practice

1. For the following problems:

   (a) find the slope of the tangent line at a general point $x = x_0$

   (b) Find the slope of the tangent line at the given x_0

   (c) Find the equation of the tangent line at the given x_0

      i. $f(x) = 3x,\ x_0 = -2$

      ii. $f(x) = x^2 - 3x,\ x_0 = 1$

      iii. $f(x) = \sqrt{x} + x\ x_0 = 1$

The Derivative Function

Recall the definition of the derivative:

We define the derivative of $f(x)$ with respect to x as

$$f' = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists.

The other forms of this definition most often used are:

$$f' = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

We can think about the derivative in several ways:

1. The instantaneous rate of change in a function at any point
2. Slope of the tangent line of a graph at any point

So this limit would exist, and therefore the derivative function, at any point where $f(x_0)$ does not have a vertical tangent line or where the sided limits are not the same. Informally, this means any x_0 that is not a corner or a point of vertical tangency has a derivative.

Point: To use the definition of the derivative, we need to calculate the limit. This generally means we need to do algebraic manipulations, like combining fractions or factoring, until the original h in the denominator of the definition cancels out.

Point: Remember that differentiability implies continuity but continuity does not imply differentiability. We can consider the function $f(x) = |x|$ at $x = 0$ to see why.

Examples

1. Find the derivative by definition of $f(x) = x^2 + 1$
   Solution:
   $$f' = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
   $$= \lim_{h\to 0} \frac{(x+h)^2 + 1 - (x^2+1)}{h}$$
   $$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + 1 - x^2 - 1}{h}$$
   $$= \lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = \lim_{h\to 0} 2x + h = 2x$$

2. Find the derivative by definition of $f(x) = \frac{1}{x}$
   Solution:
   $$f' = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
   $$= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$
   $$= \lim_{h\to 0} \frac{\frac{x}{x} \cdot \frac{1}{x+h} - \frac{1}{x} \cdot \frac{x+h}{x+h}}{h}$$
   $$= \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} = \lim_{h\to 0} \frac{-h}{hx(x+h)} = \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}x(x+h)} = \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}$$

Practice

1. For the following functions, find $f'(x)$ by definition

   (a) $y = 3x$

   (b) $y = x^2 - 2x + 5$

   (c) $y = \dfrac{1}{\sqrt{x}}$

Basic Laws of Differentiation

We start by considering a horizontal line. If we consider the slope of the floor, we think of it as flat with no slope. This means that a horizontal line has no slope.

For any real constant c: $\dfrac{d}{dx}c = 0$

Now we will work with our variable x to a power. We can think of the rule as pulling the power down, then decreasing the power by 1. Keep in mind that a constant means the variable is to the zero power

For any $n \neq -1$: $\dfrac{d}{dx}x^n = nx^{n-1}$

Examples

1. $\dfrac{d}{dx}x = 1 \cdot x^{1-1} = 1 \cdot x^0 = 1$

2. $\dfrac{d}{dx}x^3 = 3x^{3-1} = 3x^2$

3. $\dfrac{d}{dx}\sqrt{x} = \dfrac{d}{dx}x^{1/2} = \dfrac{1}{2}x^{1/2-1} = \dfrac{1}{2}x^{-1/2}$

4. $\dfrac{d}{dx}\left[\frac{1}{x^4}\right] = \dfrac{d}{dx}x^{-4} = -4x^{-4-1} = -4x^{-5}$

Most typical problems however are not single variables or constants, but polynomials with various terms with coefficients and powers. In order to solve those, we need the following two rules.

Let $f(x)$ and $g(x)$ be differentiable, and let c be any real number:

1. $\frac{d}{dx}cf(x) = c\frac{d}{dx}f(x)$
2. $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

This will allow us to now tackle a variety of problems.

Examples

1. $\frac{d}{dx}5x^4 = 5\frac{d}{dx}x^4 = 5 \cdot 4x^{4-1} = 20x^3$

2. $\frac{d}{dx}[x^7 + 3x^2] = \frac{d}{dx}x^7 + 3\frac{d}{dx}x^2 = 7x^6 + 3 \cdot 2x = 7x^6 + 6x$

Practice

1. For the following functions, find $\frac{dy}{dx}$:

   (a) $y = 7x$

   (b) $y = 4x^3 - 5x^2 + x - 12$

   (c) $y = x^\pi$

   (d) $y = \frac{1}{\sqrt{x}}$

2. For the following functions, find $y'(1)$:

(a) $y = 8$

(b) $y = -6x^3$

(c) $y = (x+3)(x-1)$

3. For the following functions, find $\dfrac{d^2y}{dx^2}$:

(a) $y = 5x^6$

(b) $y = 3x^7 - 3x^5 + 4x^3 - 2x + 1$

(c) $y = x^{5/3} + \sqrt{x} - \dfrac{1}{x^2}$

Product and Quotient Rule

Though tempting to just derive the product or quotient all at once, the general rule of thumb for derivatives is this: **Only derive one piece at a time**. This holds for product rule, quotient rule, and chain rule. This leads us to the rules for this section:

1. $\dfrac{d}{dx}[f(x)g(x)] = g(x)\dfrac{d}{dx}[f(x)] + f(x)\dfrac{d}{dx}[g(x)]$

2. $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)\dfrac{d}{dx}[f(x)] - f(x)\dfrac{d}{dx}[g(x)]}{[g(x)]^2}$

It can be helpful to think of the rules in terms of what they are doing.
For the product rule: *Derive the first, leave the second, plus derive the second, leave the first.*
For the quotient rule: *Derive the top, leave the bottom, minus leave the top derive the bottom, all over the bottom squared.*

Keep in mind that the top is the numerator and the bottom is the denominator, but it does not have the same sort of helpful cadence.

Examples

1. $\dfrac{d}{dx}\left[(x^2+7)(4x^5-8x^3)\right]$

$$= (4x^5-8x^3)\frac{d}{dx}[x^2+7] + (x^2+7)\frac{d}{dx}[4x^5-8x^3]$$

$$= (4x^5-8x^3)(2x) + (x^2+7)(20x^4-24x^2)$$

2. $\dfrac{d}{dx}\left[(3x^7-2x-1)\left(\sqrt{x}+\dfrac{2}{x}\right)\right]$

$$= \left(\sqrt{x}+\frac{2}{x}\right)\frac{d}{dx}[3x^7-2x-1] + (3x^7-2x-1)\frac{d}{dx}\left[\sqrt{x}+\frac{2}{x}\right]$$

$$= \left(\sqrt{x}+\frac{2}{x}\right)(21x^6-2) + (3x^7-2x-1)\left(\frac{1}{2\sqrt{x}}-\frac{2}{x^2}\right)$$

3. $\frac{d}{dx}\left[\frac{-2x^3+6x}{9x^2+5x-1}\right]$

$$= \frac{(9x^2+5x-1)\frac{d}{dx}[-2x^3+6x] - (-2x^3+6x)\frac{d}{dx}[9x^2+5x-1]}{(9x^2+5x-1)^2}$$

$$= \frac{(9x^2+5x-1)(-6x^2+6) - (-2x^3+6x)(18x+5)}{(9x^2+5x-1)^2}$$

4. $\frac{d}{dx}\left[(x^2+1)\left(\frac{x+5}{4x^2-1}\right)\right]$

$$= \left(\frac{x+5}{4x^2-1}\right)\frac{d}{dx}[x^2+1] + (x^2+1)\frac{d}{dx}\left[\frac{x+5}{4x^2-1}\right]$$

$$= \left(\frac{x+5}{4x^2-1}\right)(2x) + (x^2+1)\frac{(4x^2-1)\frac{d}{dx}[x+5] - (x+5)\frac{d}{dx}[4x^2-1]}{(4x^2-1)^2}$$

$$= \left(\frac{x+5}{4x^2-1}\right)(2x) + (x^2+1)\frac{(4x^2-1)(1) - (x+5)(8x)}{(4x^2-1)^2}$$

Notice in the last example how its a product rule, where one of the functions is a quotient. We use whatever rule we need at the time to derive what we have. A problem can have multiple rules in it, so it is important to use what derivative rule you need when you need to use it.

Practice

1. For the following functions, find $\dfrac{dy}{dx}$:

   (a) $y = 8x(3x^2 - 12)$

   (b) $y = (7x^4 - 3x^2 + 3x - 1)\left(2\sqrt{x} - \dfrac{9}{x^3}\right)$

   (c) $y = \dfrac{7x^3 + 2x}{1 - x^2}$

   (d) $y = \dfrac{6x}{x^2 + 2x + 12}$

2. For the following function, find all locations where the tangent line is horizontal:
$$y = \frac{x^2 + 8}{x + 1}$$

Derivatives as Rates of Change

Learning Objectives:

- Understand the connection between derivatives instantaneous rate of change.
- Apply derivatives in a rectilinear motion situation.
- Understand the meaning of marginal cost, marginal revenue, and marginal profit as derivatives.

One of derivative's most common interpretation is that it is the instantaneous rate of change of a function.

Problem 1

1. Given the function $f(x) = x^2 + 4$, find the average rate of change,
$$m_{ave} = \frac{f(b) - f(a)}{b - a},$$
when x changes from
   a. 1 to 3

   b. 1 to 1.5

   c. 1 to 1.1

   d. 1 to $1 + h$

As we see in part d, if we take the limit as h approaches 0, we have the definition of derivatives.

Definition: Instantaneous Rate of Change
Given a function $f(x)$ that is differentiable at $x = a$, the **instantaneous rate of change** of $f(x)$ at $x = a$ is given by $$m_{inst} = f'(a).$$

The motion of an object along a straight line demonstrates this idea.

Definition: Velocity, Speed, and Acceleration
Let $s(t)$ be the position function of an object at time t. The **velocity** of the object at time t is given by $v(t) = s'(t)$. The **speed** of the object at time t is given by $\lvert v(t)\rvert$. The **acceleration** of the object at time t is given by $a(t) = v'(t) = s''(t)$.

Problems 2 – 3

2. In the game of Spud, a player throws the game straight up and call out a number. The player with the number called has to catch the ball, while everyone else run away. Supposed the ball has the position function $s(t) = -16t^2 + 48t$, t is in term of second.
   a. What is the average velocity when t goes from 1 sec to 2 sec?

   b. What is the instantaneous velocity of the ball when $t = 1$?

   c. What is the speed of the ball when it hits the ground?

3. Supposed the position of an object moving along an axis is given by

$$s(t) = t^3 - 12t^2 + 45t - 50,$$

where t is in term of second.

a. Find the velocity function.

b. Determine when is the object at rest.

c. Find the acceleration function.

d. Determine when is the object neither speeding up or down.

Another common use of instantaneous rate of change is in the fields of business and economics, where we are concerned with how fast the cost, revenue, and/or profit are changing.

Definition: Marginal Cost, Revenue, Profit
If $C(x)$ is the cost to produce x units, then the **marginal cost** is $C'(x)$. If $R(x)$ is the revenue obtained from selling x units, then the **marginal revenue** is $R'(x)$. If $P(x) = R(x) - C(x)$ is the profit from selling x units, then the **marginal profit** is $P'(x)$.

Problem 4

4. Suppose the cost to produce x cups of coffee is giving by the function

$$C(x) = 0.25x^2 + 5.$$

a. If the coffee is sold at $3 per cup, determine the revenue function and profit function.

b. Determine the marginal cost, marginal revenue, and marginal profit function.

c. Determine when are the revenue equals to the cost (this is called the break-even quantity). What are the marginal profits at these quantities? Interpret your answer.

Practice Problems

Derivatives as Rates of Change

1. Find the average rate of change of the given function on the given interval.
   a. $f(x) = x^2$ on $[0, 4]$

   b. $g(x) = \sqrt{x - 3}$ on $[4, 7]$

   c. $h(x) = \frac{1}{x}$ on $[1, 2]$

2. Find the instantaneous rate of change of the given function at the given x-value.
   a. $f(x) = x^2 + 8$ at $x = 2$

   b. $f(x) = \sqrt{x}$ at $x = 4$

   c. $f(x) = \frac{1}{x^3}$ at $x = 3$

3. A ball is thrown with the starting height of 6 feet straight into the air. Its position in feet from the ground after t seconds is given by $s(t) = -16t^2 + 29t + 6$.
   a. What is the average velocity of the ball from $t = 0$ to $t = 1$?

   b. What is the instantaneous velocity of the ball at $t = 0.5$ second?

   c. What is the speed of the ball at $t = 1.5$ second?

   d. When does the ball hit the ground? What is the velocity of the ball at that instant?

4. A bottle rocket is launched from the ground, and its position in feet from the ground after t seconds is given by $s(t) = -16t^2 + 128t$. When does the rocket reach its maximum height? What is the velocity at that instant?

5. The record for the world's fastest lift is held by the Lotte World Tower in Seoul, South Korea. The double-decker lift, called Sky Shuttle, is 496 m (1,627 ft). This lift can travel from the basement to the 121^{st} floor (the observation deck) in just one minute.
a. What is the average velocity per second for the lift to go from the basement to the 121^{st} floor?

b. Suppose the position for the lift in term of second is given by $s(t) = 128.4\sqrt[3]{t}, 0 \leq t \leq 60$. Determine the velocity of the lift when it is 300m high.

6. In order to produce x units of a certain item, it costs (in hundred dollars) $C(x) = 10 + \sqrt{x}$.
a. If the item is sold for $60, what are the revenue function and profit function? Be careful with the unit.

b. Determine the marginal cost, marginal revenue, and marginal profit function.

c. Find the marginal cost of selling 1 more unit, when you sell 20 units. Compare it with the actual increase in cost going from selling 20 units to 21 units.

d. Determine the number of units sold when the cost equals the revenue. (Consider using the substitution $r = \sqrt{x}$ to turn it into a quadratic equation.) What is the marginal profit at that instant?

7. A bicycle store sold 20 bicycles When it is sold at $400. The store determined that for every $5 discounted, they can sell 1 more bicycle. Let x be the number of bicycles sold.
   a. If the bicycle costs $200 each, find the cost, revenue, and profit functions.

b. Determine the marginal cost, marginal revenue, and marginal profit function.

c. Find the marginal revenue of selling 1 more bicycle, when you sell 25 bicycles. Compare it with the actual increase in cost going from selling 25 bicycles to 26 bicycles.

d. Determine the number of bicycles sold when the cost equals the revenue. What is the marginal profit at that instant?

Answers to problems:

1a) 4 1b) 2.5 1c) 2.1 1d) 2 + h

2a) 0 2b) 16 2c) 48

3a) $v(t) = 3t^2 - 24t + 45$ 3b) $x = 3$ and $x = 5$ 3c) $a(t) = 6t - 24$ 3d) $x = 4$

4a) $R(x) = 3x,\ P(x) = 3x - 0.25x^2 - 5$

4b) $C'(x) = 0.5x,\ R'(x) = 3,\ P'(x) = 3 - 0.5x$

4c) $x = 2$ and $x = 10$, $P'(2) = 2, P'(10) = -2$. Profit is increasing at $2 per cup when it was 2 cups. Profit is decreasing at $2 per cup when it was 10 cups.

Answers to practice problems:

1a) 4 1b) 1/4 1c) -1/2

2a) 4 2b) 1/4 2c) -1/27

3a) 13 3b) 13 3c) 19 3d) t = 2, v(2) = -3

4) $t = 4, v(4) = 0$

5a) 124/15 ≈ $8.27 m/sec$ 5b) 7.84 m/sec

6a) $R(x) = 0.6x,\ P(x) = 0.6x - 10 - \sqrt{x}$

6b) $C'(x) = \frac{1}{2\sqrt{x}}$, $R'(x) = 0.6$, $P'(x) = 0.6 - \frac{1}{\sqrt{x}}$

6c) 0.1118 and 0.1104

6d) 100/9 ≈ 11, $P'(11) \approx 0.298$

7a) $C(x) = 200x,\ R(x) = x(400 - 5(x - 20)) = 500x - 5x^2,\ P(x) = 300x - 5x^2$

7b) $C'(x) = 200,\ R'(x) = 500 - 10x,\ P'(x) = 300 - 10x$

7c) $R'(25) = 250$, actual change is 245

7d) x = 60, $P'(60) = -300$

Derivatives of Trigonometric Functions

Learning Objectives:

- Be able to take the derivatives of sine and cosine functions.
- Understand how to get the derivatives for the other four trigonometric functions from sine and cosine functions.
- Understand the pattern for higher derivatives of sine and cosine functions.

We now generalize the idea of area to functions that aren't nonnegative.

Theorem: Derivatives of sine and cosine functions
The derivative of $y = \sin x$ and $y = \cos x$ are $$\frac{d}{dx}(\sin x) = \cos x\,, \frac{d}{dx}(\cos x) = -\sin x.$$

Problems 1 - 2

1. Find the derivative of $f(x) = x^2 \sin x$.

2. Find the derivative of $g(x) = \frac{\cos x}{x^3}$.

Recall that $\tan x = \frac{\sin x}{\cos x}$, $\sec x = \frac{1}{\cos x}$, $\cot x = \frac{\cos x}{\sin x}$, and $\csc x = \frac{1}{\sin x}$. Therefore, you can find the derivatives of them using quotient rules.

Problems 3 – 4

3. Find the derivative of $f(x) = \tan x$ and $g(x) = \cot x$.

4. Find the derivative for $f(x) = \sec x$ and $g(x) = \csc x$.

Theorem: Derivatives of trigonometric functions

$$\frac{d}{dx}(\tan x) = \sec^2 x\,, \qquad \frac{d}{dx}(\cot x) = -\csc^2 x,$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x\,, \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x.$$

Notice that the ones that start with a c are the ones that are negative when we take the derivative.

Problems 5 – 8

5. Find the equation of the tangent line to $f(x) = x^2 \tan x$ at $(\pi, 0)$.

6. Find the derivative of $y = \cot x \csc x$.

7. Find $y', y'', y''', y^{(4)}, y^{(5)}, y^{(6)}$, and $y^{(7)}$ for $y = \sin x$.

8. Find $y', y'', y''', y^{(4)}, y^{(5)}, y^{(6)}$, and $y^{(7)}$ for $y = \cos x$.

Notice the pattern as you take the derivatives, it repeats after taking the derivatives 4 times.

Theorem
For any whole number n, $$\frac{d^n}{dx^n}(\sin x) = \frac{d^{n+4}}{dx^{n+4}}(\sin x),$$ and $$\frac{d^n}{dx^n}(\cos x) = \frac{d^{n+4}}{dx^{n+4}}(\cos x).$$

Thus, given any order derivatives of sine or cosine, we can divide the order by 4 and see what the remainder is.

Problems 9 – 10

9. Find the 29th derivative of $y = \sin x$.

10. Find the 74th derivative of $y = \cos x$.

Practice Problems

1. Find the derivatives for the following functions:
   a. $f(x) = 2\sin x + 4\cos x$

   b. $g(x) = \cot x - 4\cos x$

   c. $h(x) = 2\csc x - 7\sec x$

   d. $y = \sin x \tan x$

e. $y = x^3 \csc x$

f. $y = \frac{x+\csc x}{1+\sin x}$

2. Find the following higher order derivatives:
   a. $\frac{d^2}{dx^2}(\tan x)$

b. $\frac{d^2}{dx^2}(\sec x)$

c. $\frac{d^{101}}{dx^{101}}(\sin x)$

d. $\frac{d^{26}}{dx^{26}}(\cos x)$

e. $\frac{d^{2024}}{dx^{2024}}(7\sin x - 3\cos x)$

Answers to problems:

1) $f'(x) = x^2 \cos x + 2x \sin x$

2) $g'(x) = \frac{-x^2 \sin x - 3 \cos x}{x^4}$

3 and 4) check the theorem on the next page.

5) $y = \pi^2 x - \pi^3$

6) $y' = -\csc x\,(\cot^2 x + \csc^2 x)$

7) $y' = \cos x\,, y'' = -\sin x\,, y''' = -\cos x\,, y^{(4)} = \sin x,$

$y^{(5)} = \cos x\,, y^{(6)} = \sin x\,, y^{(7)} = \cos x$

8) $y' = -\sin x\,, y'' = -\cos x\,, y''' = \sin x\,, y^{(4)} = \cos x,$

$y^{(5)} = -\sin x\,, y^{(6)} = -\cos x\,, y^{(7)} = -\sin x$

9) $y^{(29)} = \cos x$

10) $y^{(74)} = -\cos x$

Practice problems:

1a) $f'(x) = 2\cos x - 4\sin x$

1b) $4\sin x - \csc^2 x$

1c) $h'(x) = -2\csc x \cot x - 7\sec x \tan x$

1d) $y' = \cos x \tan x + \sin x \sec^2 x = \sin x + \tan x \sec x$

1e) $y' = x^2 \csc x\,(3 - x\cot x)$

1f) $y' = \frac{1 + \sin x - \cos x - 2\cot x - \csc x \cot x}{(1 + \sin x)^2}$

2a) $2\tan x \sec^2 x$

2b) $\sec x\,(\sec^2 x + \tan^2 x)$

2c) $\cos x$

2d) $-\cos x$

2e) $7\sin x - 3\cos x$

The Chain Rule

Let g be a differentiable function at x and f be a differentiable function at $g(x)$, then the derivative of the composite function $(f \circ g)(x) = f(g(x))$ is

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))g'(x)$$

Alternatively, if y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Problem-Solving Strategy:

1. *To differentiate $f(g(x))$, begin by identifying the outer function $f(x)$ and the inner function $g(x)$*
2. *Find $f'(x)$ and evaluate it at $g(x)$ to get $f'(g(x))$*
3. *Find $g'(x)$*
4. *Write the expression for $f'(g(x))g'(x)$*

Examples

Example 1 Find the derivative of $h(x) = (13x^2 - x)^5$.

If we let $g(x) = 13x^2 - x$ then $h(x) = (13x^2 - x)^5 = [g(x)]^5$. By the Chain Rule $h'(x) = 5\,[g(x)]^4\,g'(x)$.

$g(x) = 13x^2 - x$, and $g'(x) = 26x - 1$. Hence $h'(x) = 5\left[13x^2 - x\right]^4(26x - 1)$.

Example 2 Find the equation of the tangent line to the graph of $h(x) = \cos^3(\pi x)$ at $x = \frac{1}{4}$.

We can rewrite $h(x) = \cos^3(\pi x)$ as $h(x) = (\cos(\pi x))^3$.

By the Chain Rule we have $h'(x) = 3\left(\cos(\pi x)\right)^2 \dfrac{d}{dx}\left(\cos(\pi x)\right)$.

After we apply the Chain Rule one more time we get $h'(x) = 3\left(\cos(\pi x)\right)^2 \sin(\pi x)\dfrac{d}{dx}(\pi x)$.

Hence, $h'(x) = 3\pi\left(\cos(\pi x)\right)^2 \sin(\pi x)$.

The slope of the tangent line is $h'\left(\frac{1}{4}\right) = 3\pi\left(\cos\left(\frac{1}{\pi}\right)\right)^2 \sin\left(\frac{1}{\pi}\right) = 3\pi\left(\dfrac{\sqrt{2}}{2}\right)^2 \dfrac{\sqrt{2}}{2} = \dfrac{3\sqrt{2}\,\pi}{4}$.

The equation of the tangent line is $y - h\left(\dfrac{1}{4}\right) = \dfrac{3\sqrt{2}\,\pi}{4}\left(x - \dfrac{1}{4}\right)$.

Alternatively, the line equation is $y - \dfrac{\sqrt{2}}{4} = \dfrac{3\sqrt{2}\,\pi}{4}\left(x - \dfrac{1}{4}\right)$.

Exercises

1. Find the derivative of each function.

   (a) $y = \cos^5 x$.

   (b) $y = \cos x^5$

2. Find the derivative of each function.

   (a) $y = (2x^3 - x + 6 + x^{-1})^3$.

   (b) $y = \sqrt[3]{2x^3 - x + 6 + x^{-1}}$

3. Find the derivative of each function.

(a) $y = (\tan x + \sin x)^{-3}$

(b) $y = \tan^{-3} x + \sin^{-3} x$

4. Find the derivative of each function.

(a) $y = (\tan x + \sin x)^{-3}$

(b) $y = \tan^{-3} x + \sin^{-3} x$

5. Find the derivative of the function $y = \dfrac{\pi}{\sin^2 x}$.

6. Find the derivative of each function.

   (a) $y = \sqrt{6 + \sec(\pi x^2)}$

   (b) $y = 6 + \sqrt{\sec(\pi x^2)}$

7. Find the equation of the tangent line to the graph of the function $y = \dfrac{\pi}{\sin^2 x}$ at $x = \frac{\pi}{2}$.

8. The position function of a freight train is given by $s(t) = 100(t+1)^{-2}$, with s in meters and t in seconds. At time $t = 6$ s, find the trains

   (a) velocity and

   (b) acceleration

   (c) Is the train speeding up or slowing down? Use your answer from the two previous questions.

9. The total cost to produce x boxes of Thin Mint Girl Scout cookies is C dollars, where $C(x) = 0.0001x^3 - 0.02x^2 + 3x + 300$. In t weeks production is estimated to be $x(t) = 1600 + 100t$ boxes.

(a) Find the marginal cost $C'(x)$.

(b) Use Leibnitz's notation for the Chain Rule, $\frac{dC}{dt} = \frac{dC}{dx}\frac{dx}{dt}$, to find the rate with respect to time t that the cost is changing.

(c) Use your previous answer to determine how fast costs are increasing when $t = 2$ weeks. Do not forget to include units in your answer.

Derivatives of Inverse Functions.

Inverse Function Theorem:

Let $f(x)$ be a function that is both invertible and differentiable. Let $y = f^{-1}(x)$ be the inverse of $f(x)$. For all x satisfying $f'\left(f^{-1}\right) \neq 0$,

$$\frac{dy}{dx} = \frac{d}{dx}\left(f^{-1}(x)\right) = \left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

Alternatively, if $y = g(x)$ is the inverse of $f(x)$, then

$$g'(x) = \frac{1}{f'\left(g(x)\right)}.$$

Examples

Example 1. *Use the inverse function theorem to find the derivative of $g(x) = \dfrac{1}{x+2}$. Compare the result obtained by differentiating $g(x)$ directly.*

The inverse of $g(x) = \dfrac{1}{x+2}$ is $f(x) = \dfrac{1-2x}{x} = \dfrac{1}{x} - 2$. Since $g'(x) = \dfrac{1}{f'\left(g(x)\right)}$, begin by finding $f'(x)$. Thus,

$$f'(x) = \frac{-1}{x^2} \text{ and } f'\left(g(x)\right) = \frac{-1}{\left(g(x)\right)^2} = \frac{-1}{\left(\frac{1}{x+2}\right)^2} = -(x+2)^2.$$

Finally,

$$g'(x) = \frac{1}{f'\left(g(x)\right)} = \frac{-1}{(x+2)^2}.$$

We can verify that this is the correct derivative by applying the quotient rule to $g(x)$ to obtain

$$g'(x) = \frac{-1}{(x+2)^2}.$$

Example 2. *For the function $f(x) = x^3 + 2x + 3$ find the equation of the tangent line to its inverse function f^{-1} at the point $P(3,0)$.*

We begin by finding the slope of the tangent line, which is equal to the derivative of the inverse function evaluated at $x = 3$. By the Inverse Function Theorem

$$\left(f^{-1}\right)'(3) = \frac{1}{f'\left(f^{-1}(3)\right)}.$$

Differentiating $f(x)$ we get $f'(x) = 3x^2 + 2$. Given the coordinates of the point P we conclude that $f^{-1}(3) = 0$. Putting it all together we get

$$m = \left(f^{-1}\right)'(3) = \frac{1}{f'\left(f^{-1}(3)\right)} = \frac{1}{f'(0)} = \frac{1}{2}.$$

The point-slope form of the equation of the line is $y - 0 = \dfrac{1}{2}(x-3)$ or simply $y = \dfrac{1}{2}(x-3)$.

Exercises

For the following functions use the Inverse Function Theorem to find BOTH $\left(f^{-1}\right)'(x)$ *and* $\left(f^{-1}\right)'(a)$.

1. $f(x) = 2x^3 - 1$, $a = 1$
2. $f(x) = x + \sqrt{x}$, $a = 2$
3. $f(x) = x^2 + 3x + 2,\ x \geq \frac{3}{2}$, $a = 2$

For the following functions use the Inverse Function Theorem to find the equation of the tangent line o its inverse function f^{-1} *at the given point* P.

4. $f(x) = \frac{4}{1+x^2}$, $P(2,1)$
5. $f(x) = (x^3+1)^4$, $P(16,1)$
6. $-x^3 - x + 2$, $P(-8,2)$

Derivatives of Inverse Trigonometric Functions.

Derivatives of Inverse Trigonometric Functions:

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1-x^2}$$
$$\frac{d}{dx}\cot^{-1}x = \frac{-1}{1-x^2}$$
$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$$
$$\frac{d}{dx}\cos^{-1}x = \frac{-1}{|x|\sqrt{x^2-1}}$$

Examples

Example 1. *Find the derivative of* $f(x) = \tan^{-1}\left(2x^3\right)$.

Using the Chain Rule we get

$$f'(x) = \frac{1}{1+(2x^3)^2}\cdot\frac{d}{dx}\left(2x^3\right) = \frac{1}{1+4x^6}\cdot 6x^2 = \frac{6x^2}{1+4x^6}$$

Example 2. *Find the equation of the tangent line to the graph of* $f(x) = x^2 \sin^{-1} x$ *at* $x = \frac{1}{2}$.

By the applying the Product Rule, we get

$$f'(x) = 2x \cdot \sin^{-1} x + \frac{1}{\sqrt{1-x^2}} \cdot x^2$$

$$f'\left(\frac{1}{2}\right) = 2 \cdot \frac{1}{2} \cdot \sin^{-1}\left(\frac{1}{2}\right) + \frac{1}{\sqrt{1-\left(\frac{1}{2}\right)^2}} \cdot \left(\frac{1}{2}\right)^2 = \frac{\pi}{6} + \frac{2}{\sqrt{3}} \cdot \frac{1}{4} = \frac{\pi}{6} + \frac{\sqrt{3}}{6} = \frac{\pi + \sqrt{3}}{6}$$

Because $f\left(\frac{1}{2}\right) = \frac{\pi}{12}$, *the point-slope from of the equation of the tangent line is*

$$y - \frac{\pi}{12} = \frac{\pi + \sqrt{3}}{6}\left(x - \frac{1}{2}\right)$$

Exercises

For the following functions find y' *using Derivatives of Inverse Trigonometric Functions formulas.*

7. $y = \sin^{-1}\sqrt{x}$
8. $y = \sqrt{x}\ \cos^{-1} x$
9. $y = \dfrac{\sin^{-1} x}{\sqrt{x}}$
10. $\sqrt{\cos^{-1} x}$
11. $y = \dfrac{1}{\tan^{-1} x}$
12. $y = \sec^{-1}(-x)$
13. *The position of a moving hockey puck after* t *seconds is* $s(t) = \tan^{-1} x$ *where* s *is in meters.*
    (a) *Find the velocity of the hockey puck at any time* t.
    (b) *Find the acceleration of the puck at any time* t.
    (c) *Evaluate (a) and (b) for* $t = 2, 4,$ *and* 6 *seconds.*
    (d) *What conclusion can be drawn from the results in (c)?*

Implicit Differentiation

Problem-Solving Strategy:

Use the following steps to differentiate an equation that defines a function y implicitly in terms of a variable x:

1. *Apply the differentiating operator $\frac{d}{dx}$ to both sides of the equation to find derivatives. Keep in mind that whenever you see an expression involving y you need to use the Chain Rule.*

2. *Rewrite the resulting equation so that every term containing $\frac{dy}{dx}$ appear on the left side of the equation, and the rest of the terms appear on the right side.*

3. *Factor out $\frac{dy}{dx}$ on the left side.*

4. *Solve for $\frac{dy}{dx}$ by dividing both side of the equation by an appropriate algebraic factor.*

Examples

Example 1. Find $\frac{dy}{dx}$ if $x^3 + y^3 = 1$.

We will work under the assumption that y is defined implicitly by the equation.

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}(1)$$

$$\frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(y^3\right) = 0$$

$$3x^2 + 3y^2 \times \frac{dy}{dx} = 0$$

$$3y^2 \times \frac{dy}{dx} = -3x^2$$

$$\frac{dy}{dx} = -\frac{x^2}{y^2}$$

Example 2. Find the equation of the tangent line to the graph of $x^3 \sin y + y = 4x + 3$ at $(-\frac{3}{4}, 0)$.

$$\frac{d}{dx}\left(x^3 \sin y + y\right) = \frac{d}{dx}(4x + 3)$$

$$\frac{d}{dx}\left(x^3 \sin y\right) + \frac{d}{dx}(y) = 4$$

$$\left(\frac{d}{dx}(x^3) \sin y + \frac{d}{dx}(\sin y) x^3\right) = 4$$

$$3x^2 \sin y + \left(\cos y \times \frac{dy}{dx}\right) x^3 + \frac{dy}{dx} = 4$$

$$x^3 \cos y \frac{dy}{dx} + \frac{dy}{dx} = 4 - 3x^2 \sin y$$

$$\frac{dy}{dx}\left(x^3 \cos y + 1\right) = 4 - 3x^2 \sin y$$

$$\frac{dy}{dx} = \frac{4 - 3x^2 \sin y}{x^3 \cos y + 1}$$

The slope of the tangent line is $\frac{dy}{dx}|_{(-\frac{3}{4},0)} = \frac{4 - 3\left(-\frac{3}{4}\right)^2 \sin 0}{\left(-\frac{3}{4}\right)^3 \cos 0 + 1} = \frac{256}{37}$

The tangent line equation is $y - 0 = \frac{256}{37}\left(x - \left(-\frac{3}{4}\right)\right)$.

Alternatively, the equation can be written as $y = \frac{256}{37}\left(x + \frac{3}{4}\right)$.

Exercises

1. Use implicit differentiation to find both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

   (a) $3x^2 - 2y^3 = 4$.

   (b) $x^2y - 2y^3 = 4$

2. Use implicit differentiation to find $\frac{dy}{dx}$.

(a) $x - \cos y = 1$.

(b) $xy - \sin y = 1$.

(c) $x - \cos(xy) = 1$.

3. Use implicit differentiation to find $\frac{dy}{dx}$.

(a) $-y - 2x = \frac{x}{7}$

(b) $-yx - 2 = \frac{x}{7}$

4. Find the equation of the tangent line to the graph of the equation $xy + \sin x = 1$ at the point $(2, 1)$.

5. Find the equation of the tangent line to the graph of the equation $\tan^{-1}(x + y) = x^2 + \frac{\pi}{4}$ at the point $(0, 1)$.

6. The number of cell phones produced when x dollars is spent on labor and y dolars is pent on capital invested by a manufacturer can be modeled by the equation $60x^{3/4}y^{1/4} = 3240$.

   (a) Find $\frac{dy}{dx}$ and evaluate at the point $(81, 16)$.

   (b) Interpret the above result.

Derivatives of Exponential Functions.

Derivative of the Natural Exponential Function.

Let $f(x) = e^x$ be the natural exponential function. Then

$$f'(x) = \frac{d}{dx}(e^x) = e^x.$$

In general,

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)} \cdot g'(x).$$

Derivative of the General Exponential Function.

Let $h(x) = b^x$, then

$$h'(x) = b^x \ln b.$$

More generally,

$$\frac{d}{dx}\left(b^{g(x)}\right) = b^{g(x)} \cdot g'(x) \ln b.$$

Examples

Example 1. If $A(t) = 1000e^{0.3t}$ describes the mosquito population after t days, what is the rate of change of $A(t)$ after 4 days?

First we find $A(t)$ using the Chain Rule.

$$A'(t) = 1000e^{0.3t} \cdot \frac{d}{dt}(0.3t) = 300e^{0.3t}$$

After 4 days the rate of change of the mosquito population is

$$A'(4) = 300e^{0.3 \cdot 4} \approx 338 \text{ mosquitos per day}$$

Note that the ratio of the rate of change of the population, $A'(t)$, to the population, $A(t)$ stays constant for all t.

$$\frac{A'(t)}{A(t)} = \frac{300e^{0.3t}}{1000e^{0.3t}} = 0.3$$

Example 2. Find the slope of the tangent line to $h(x) = \dfrac{3^x}{3^x + 2}$ at $x = 0$.

Combining the Quotient Rule together with the Derivative of the General Exponential Function we have

$$h'(x) = \frac{3^x \ln 3(3^x + 2) - 3^x \ln 3(3^x)}{(3^x + 2)^2} = \frac{2 \cdot 3^x \ln 3}{(3^x + 2)^2}.$$

the slope of the tangent line at $x = 0$ is

$$h'(0) = \frac{2 \cdot 3^0 \ln 3}{(3^0 + 2)^2} = \frac{2 \ln 3}{9}$$

Exercises

For the following exercises, find $f'(x)$ for each function.

1. $f(x) = x^3 e^x$
2. $f(x) = x^3\, 3^x$
3. $f(x) = \dfrac{x}{e^{-x}}$
4. $f(x) = \sqrt{e^{4x} + 3x}$
5. $f(x) = 5^{\cos 4x} + x$
6. Find the equation of the tangent line to the graph of $f(x) = 4x\, e^{x^2-1}$ at the point where $x = -1$.
7. The number of cases of influenza in New York City from the beginning of 1960 to the beginning of 1961 is modeled by the function $N(t) = 5.3\, e^{0.093t^2 - 0.87t}$, 0, where $N(t)$ gives the number of cases (in thousands) and t is measured in years, with $t = 0$ corresponding to the beginning of 1960.
   (a) Show work that evaluates $N(0)$ and $N(4)$. Briefly describe what these values indicate about the disease in New York City.
   (b) Show work that evaluates $N'(0)$ and $N'(3)$. Briefly describe what these values indicate about the disease in New York City.

Derivatives of Logarithmic Functions.

Derivative of the Natural Logarithmic Function

If $x > 0$ and $y = \ln x$, then

$$y' = \frac{d}{dx}(\ln x) = \frac{1}{x}$$

More generally, let $g(x)$ be a differentiable function. For all values of x for which $g(x) > 0$, the derivative of $h(x) = \ln(g(x))$ is given by

$$h'(x) = \frac{1}{g(x)} \cdot g'(x) = \frac{g'(x)}{g(x)}.$$

Derivative of the General Logarithmic Function

If $y = \log_b x$, then

$$y' = \frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}.$$

More generally, if $h(x) = \log_b(g(x))$, then for all values of x forwhich $g(x) > 0$,

$$h'(x) = \frac{g'(x)}{g(x) \ln b}.$$

Examples

Example 3. Differentiate: $f(x) = \ln\left(x^3 + x - 7\right)$.

Using the Chain Rule we get

$$f'(x) = \frac{1}{x^3 + x - 7} \cdot \frac{d}{dx}\left(x^3 + x - 7\right) = \frac{1}{x^3 + x - 7} \cdot \left(3x^2 + 1\right) = \frac{3x^2 + 1}{x^3 + x - 7}$$

Example 4. Find the equation of the tangent line to the graph of $f(x) = \log_2(\sin x)$ at $x = \frac{\pi}{4}$.

Combining the Chain Rule and the Derivative of the General Logarithmic Function formula we get

$$f'(x) = \frac{1}{\sin x \, \ln 2} \cdot \frac{d}{dx}(\sin x) = \frac{\cos x}{\sin x \, \ln 2} = \frac{\cot x}{\ln 2}$$

The slope of the tangent line at $x = \frac{\pi}{4}$ is

$$f'\left(\frac{\pi}{4}\right) = \frac{\cot\left(\frac{\pi}{4}\right)}{\ln 2} = \frac{1}{\ln 2}.$$

The value of the function at $x = \frac{\pi}{4}$ is $-\frac{1}{2}$. (Verify!) And the equation of the tangent in point-slope line is

$$y = \frac{1}{\ln 2}\left(x - \frac{\pi}{4}\right).$$

Exercises

For the following exercises, find $f'(x)$ for each function.

8. $f(x) = \ln\sqrt{5x - 7}$
9. $f(x) = \log_2\sqrt{5x - 7}$
10. $f(x) = \sqrt{\ln(5x - 7)}$
11. $f(x) = \ln(5\sqrt{x} - 7)$
12. $f(x) = 2^x \log_3 7^{x^2 - 1}$

Logarithmic Differentiation.

Problem-Solving Stratefy: Using Logarithmic Differentiation

(a) To differentiate $y = h(x)$ using logarithmic differentiation, take the natural logarithm of both sides of the equation to obtain $\ln y = \ln(h(x))$.

(b) Use properties of logarithms to expand $\ln(h(x))$ as much as possible.

(c) Differentiate both sides of the equation. On the left we will have $\frac{1}{y}\frac{dy}{dx}$.

(d) Multiply both sides of the equation by y to solve for $\frac{dy}{dx}$.

(e) Replace y by $h(y)$.

Examples

Example 5. Find the derivative of $y = \left(x^2 + x\right)^{\cos x}$.

Using logarithmic differentiation on both sides we get

$$\ln y = \ln \left(x^2 + x\right)^{\cos x}$$

$$\ln y = \cos x \ln \left(x^2 + x\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \sin x \ln \left(x^2 + x\right) + \cos x \left(\frac{2x+1}{x^2+x}\right)$$

$$\frac{dy}{dx} = y \left[\sin x \ln \left(x^2 + x\right) + \cos x \left(\frac{2x+1}{x^2+x}\right)\right]$$

$$\frac{dy}{dx} = \left(x^2 + x\right)^{\cos x} \left[\sin x \ln \left(x^2 + x\right) + \cos x \left(\frac{2x+1}{x^2+x}\right)\right]$$

Exercises

For the following exercises, use logarithmic differentiation to find $\frac{dy}{dx}$.

13. $y = x^{\sqrt{x}}$
14. $y = (\sqrt{x})^{x}$
15. $y = (\sin 2x)^{4x}$
16. $y = x^{\log_2 x}$

CHAPTER 3: APPLICATIONS OF DERIVATIVES

Now that you have learned how to find derivatives for all types of functions, you can now begin to understand how the derivative is used in the context of mathematics and the real-world.

Learning Objectives:

1. Solve problems involving quantities that change with respect to time.
2. Approximate the value of an irrational number using the linearization of a function.
3. Determine the precise locations where a function has relative extrema and classify such extrema using the First Derivative Test or Second Derivative Test.
4. Determine the precise locations where a function has points of inflection.
5. Sketch the graph of a polynomial function or rational function by studying its original form, first derivative, and second derivative.
6. Find limits of functions with indeterminate forms by using L'Hôpital's Rule.
7. Approximate the location of a root of a function using Newton's Method.
8. Find the antiderivative of a function.
9. Solve an initial-value problem based on a first derivative or based on a second derivative.

Related Rates

Learning Objectives:

- Understand the concept of related rates.
- Use related rates in an application problem.

One of implicit differentiation's applications is using the rate of change of a variable to determine how a related variable is changing.

Problem 1

1. Suppose $x(t)$ and $y(t)$ are related by the equation $x^2 - y = 5$. If $\frac{dx}{dt} = 4$ when $x = 3$, find $\frac{dy}{dt}$.
   a. Implicitly differentiate the equation with respect to t.

   b. Solve for dy/dt.

   c. Substitute $\frac{dx}{dt} = 4$ and $x = 3$ into the equation.

In applications, you will need to use the information given to come up with the relation.

Strategy
1. Assign symbols to all variables involved in the problem. Draw a figure if applicable. 2. State, in terms of the variables, the information that is given and the rate to be determined. 3. Find an equation relating the variables introduced in step 1. 4. Using the chain rule, differentiate both sides of the equation found in step 3 with respect to the independent variable. This new equation will relate the derivatives. 5. Substitute all known values into the equation from step 4, then solve for the unknown rate of change.

Problems 2 – 3

2. A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius r of the outer ripple is increasing at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area A of the disturbed water changing?
   a. What are the variables involved in the problem? Draw a figure.

   b. What is the equation that relates the variables together?

   c. Solve for the unknown rate of change following the strategy.

3. How fast does the water level rise in a cylinder can of radius 2 ft if water is being poured in at a rate of 3 ft^3/sec ?

Practice Problems

1. Gas is escaping from a spherical balloon at a rate of 2 ft^3/min. How fast is the radius decreasing when the radius is 12 feet?

2. A ladder 26 feet long rests on the (horizontal) ground and leans against a (vertical) wall. The base of the ladder is pulled away from the wall at a rate of 3 ft/sec. How fast is the top of the ladder sliding down the wall when the base is 10 feet from the wall?

3. A man 6 feet tall is walking at 3 ft/sec toward a streetlight 18 feet high. How fast is the length of his shadow changing?

4. A balloon is rising straight up from a level field and is being tracked by a camera 500 feet from the point of lift off. At the instant when the camera's angle of elevation is $\pi/4$, that angle is increasing at a rate of 0.24 radians/min. How fast is the balloon rising at that instant?

5. Water is running out of a conical funnel at a rate of 1 in^3/sec. If the altitude of the funnel is 8 inches and the radius of its base is 4 inches, how fast is the water level dropping when it is 2in from the top?

Answers to problems:

1a) $2x\frac{dx}{dt} - \frac{dy}{dt} = 0$

1b) $\frac{dy}{dt} = 2x\frac{dx}{dt}$

1c) $\frac{dy}{dt} = 24$

2a) A = area of the circle, r = radius of the circle

2b) $A = \pi r^2$

2c) $\frac{dA}{dt} = 8\pi\ ft^2/sec$

3) $\frac{3}{4\pi} ft/sec$

Practice problems:

1) $\frac{1}{288\pi} ft/min$

2) -1.25 ft/sec

3) -1.5 ft/sec

4) 240 ft/min

5) $\frac{1}{18\pi}\ in/sec$

Linearization and Differentials

Summary: Linearization of a Function at $x = a$
➢ Find $f(a)$. ➢ Find $f'(a)$. ➢ Build the linearization as a tangent line: $L(x) = f(a) + f'(a)(x - a)$

Linearization is meant to be an approximation of a function near the point of tangency:

$$f(x) \approx L(x)$$

To determine the accuracy of a linearization at $x = c$, determine the percent error:

$$PE(c) = 100\% \cdot \frac{|L(c) - f(c)|}{|f(c)|}$$

Try It #1
Use a linearization of $f(x) = \sqrt{x}$ at $x = 1$ to estimate the value of $\sqrt{0.9}$. Calculate the percent error of the linearization.

(TO INSTRUCTIONAL DESIGN: CREATE SOLUTIONS LINK HERE)

Try It #2
Approximate the value of $\cos 91°$ using a linearization. Calculate the percent error of the linearization.

(TO INSTRUCTIONAL DESIGN: CREATE SOLUTIONS LINK HERE)

Summary: Differentials
➢ If $y = f(x)$, then $dy = f'(x)\,dx$. ➢ For an arbitrary dx at $x = a$, the value of dy can be determined. ➢ There is a direct variation between dy and dx at any $x = a$.

Try It: Find the value of dy for the given values of x and dx	
$f(x) = x^2 + 2x, x = 3, dx = 0.1$	$f(x) = \cos x\,, x = \frac{\pi}{3}, dx = \frac{2}{\pi}$

(TO INSTRUCTIONAL DESIGN: CREATE SOLUTIONS LINK HERE)

Try It: An Application of Differentials
The radius of a sphere is 4 inches with a potential measurement error of 0.2 inches (meaning, $-0.2 \leq dx \leq 0.2$). Determine the potential error in the volume of the sphere.

SOLUTIONS

1. Try It #1: The linearization is $L(x) = 1 + \frac{1}{2}(x - 1)$ and $L(0.9) = 0.95$. The percent error at $c = 0.9$ is approximately 0.14%.

2. Try It #2: Build a linearization at $x = 90° = \frac{\pi}{2}$, so $L(x) = 0 - 1\left(x - \frac{\pi}{2}\right) = \frac{\pi}{2} - x$ and since $x = 91° = \frac{91\pi}{180}$, $L\left(\frac{91\pi}{180}\right) = \frac{\pi}{2} - \frac{91\pi}{180} = -\frac{\pi}{180}$. The percent error at $c = -\frac{\pi}{180}$ is approximately 0.01%.

3. Try It for Differentials #1
   a. $dy = 0.8$.
   b. $dy = -\frac{\sqrt{3}}{\pi}$.

4. Try It for Differentials #2: The error for the volume of the sphere is: $-\frac{64\pi}{5} \leq dV \leq \frac{64\pi}{5}$.

Extreme Values of Functions

Steps to finding absolute extrema (maxima and minima) of a function:

1. Ensure that the function is continuous over the closed interval [a , b].

2. Find the critical points of the function within that interval. (No need to find ones outside.)

3. Plug the critical points that you found into the function, as well as the endpoints of the interval (a and b), and evaluate the function at each point.

4. The lowest function value is the ***absolute minimum***, and the highest is the ***absolute maximum***.

Try it: Find the absolute extrema of the following functions:

$$f(x) = 4x^3 - 24x^2 - 60x - 11 \text{ } on \text{ } [-5, 3]$$

Try it: Suppose that the amount of money in a bank account after t years is given by:

$$A(t) = 5000 - 5te^{3 - \frac{t^2}{4}}$$

Find the maximum and minimum amounts of money over the 15 years that the account is open.

Note that it is important, especially with word problems, to think carefully about the critical points that you found before you plug them into the function. For example, in the problem above, it would not make sense to plug in a negative value for t, since it is dealing with time.

Most of the time, the derivative will exist everywhere on the interval, so you are looking for points where the derivative equals 0. This will not always be the case. There may be times where you find critical values at points where the derivative does not exist. Keep that in mind.

Rolle's Theorem and the Mean Value Theorem (MVT)

Conditions for Rolle's Theorem

1. The function $f(x)$ must be continuous on the closed interval $[a , b]$.
2. f must be differentiable on the open interval (a , b).
3. $f(a) = f(b)$

If so, then there is some c such that $a < c < b$ and $f\,'(c) = 0$. (f has a critical point in (a , b).)

This theorem is important for two reasons: It helps to show that functions have exactly one real root on an interval, and it helps prove the stronger Mean Value Theorem (MVT).

Try It: Determine if Rolle's theorem applies to the function $f(x) = \frac{x^2-4}{x}$

Conditions for the Mean Value Theorem (MVT)

1. $f(x)$ must be continuous on the closed interval $[a , b]$.
2. $f(x)$ must be differentiable on the open interval (a , b).

If so, then there is some c such that $a < c < b$ and $f\,'(c) = \frac{f(b)-f(a)}{b-a}$.

Note the similarities to the MVT formula and the formula for slope. This makes sense because $f\,'(c)$ is the slope of the tangent line to f at c.

The MVT also tells us that slope of the secant line that connects $f(a)$ and $f(b)$ is equal to the slope of the tangent line at $x = c$.

Try It: Determine all the values c that satisfy the conclusion of the MVT for the function:

$$f(x) = x^3 + x - 1 \text{ on the interval } [0 , 2]$$

Important corollaries to the MVT

1. On the interval (a , b), if $f\,'(x) = 0$ for all of the x-values in the interval, then $f\,(x)$ is constant on (a , b).

Try It: Prove this corollary.

2. For all x in the interval (a , b), if $f\,'(x) = g\,'(x)$, then $f\,(x) = g\,(x) + c$ (c is some constant.)

Try It: Prove this corollary.

The MVT has applications to physics, as well, particularly the velocity of an object.

Try it: Suppose that you drive on I-95 for 25 miles in the toll lane. Some weeks later, you receive a speeding ticket in the mail claiming that you entered the toll lane at 2:30 PM and exited 25 miles later at 2:50 PM. If you decide to appeal the ticket, can they prove you were speeding?

Information About Functions from First and Second Derivatives

Summary: Information Obtained from $f(x)$.
➢ Intercepts. ➢ End Behavior (found using $\lim_{x \to \pm\infty} f(x)$). ➢ Points of Discontinuity (also found using limits).

Summary: Information Obtained from $f'(x)$.
➢ Critical Points. ➢ Relative Extrema. ➢ Intervals of Increase. ➢ Intervals of Decrease.

Summary: Information Obtained from $f''(x)$.
➢ Points of Inflection. ➢ Intervals of Concavity. ➢ Classification of a Critical Point.

The First Derivative Test

Let $f(x)$ have a critical point at $x = c$. Then:

- If $f'(x) < 0$ for $x < c$ and if $f'(x) > 0$ for $x > c$, then $x = c$ is a relative minimum of $f(x)$.
- If $f'(x) > 0$ for $x < c$ and if $f'(x) < 0$ for $x > c$, then $x = c$ is a relative maximum of $f(x)$.
- If $f'(x)$ does not change sign at $x = c$, then this test is inconclusive about the nature of $x = c$.

Steps: Using the First Derivative Test

1. Find $f'(x)$.
2. Solve $f'(x) = 0$ and, if necessary, see where $f'(x)$ is undefined within the domain of $f(x)$ to obtain the critical points.
3. Create a number line and mark the critical values on the number line, as shown below:

4. Select a test number from each section of the number line.
5. Evaluate $f'(x)$ at each test number and record whether the sign of $f'(x)$ is positive or negative at each test point.
6. Analyze the sign changes to classify each critical point according to the First Derivative Test.

Try It: Find and classify the critical points for the following functions.	
$f(x) = x^3 + 3x^2 - 45x + 6$	$f(x) = \frac{3}{4}(x^2 - 1)^{\frac{4}{3}}$

(TO INSTRUCTIONAL DESIGN: CREATE SOLUTIONS LINK HERE)

The Second Derivative Test

Let $x = c$ be a critical value of $f(x)$. Then:

- If $f''(c) < 0$, then $f(x)$ has a relative maximum at $x = c$.
- If $f''(c) > 0$, then $f(x)$ has a relative minimum at $x = c$.
- If $f''(c) = 0$, then this test is inconclusive about the nature of $x = c$.

Steps: Using the Second Derivative Test
1. Find $f'(x)$. 2. Solve $f'(x) = 0$ and, if necessary, see where $f'(x)$ is undefined within the domain of $f(x)$ to obtain the critical points. 3. Find $f''(x)$. 4. Substitute each critical value into $f''(x)$ and draw a conclusion based on the Second Derivative Test

Try It: Find and classify the critical points for the following functions.	
$f(x) = \frac{2}{5}x^5 + \frac{1}{2}x^4 - \frac{16}{3}x^3 - 12x^2 - 4$	$f(x) = \tan^{-1}\left(\frac{x}{x^2 + 1}\right)$

(TO INSTRUCTIONAL DESIGN: CREATE SOLUTIONS LINK HERE)

The Concavity Test

The property of **<u>concavity</u>** describes the bending direction of a curve: does the curve have an upward bend or a downward bend?

Let $f(x)$ be continuous on $[a, b]$ and twice differentiable on (a, b). Then:

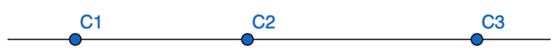
- If $f''(x) < 0$ for all $a < x < b$, then $f(x)$ is **<u>concave down</u>** on (a, b).
- If $f''(x) > 0$ for all $a < x < b$, then $f(x)$ is **<u>concave up</u>** on (a, b).

If $f''(x) = 0$ or $f''(x)$ is undefined at some point $x = c$, then its possible that the concavity of $f(x)$ will change at $x = c$, and thus we would call $x = c$ a **<u>point of inflection</u>**.

The concavity test tells us that if $f''(x)$ changes sign at $x = c$, then $f(x)$ has a point of inflection at $x = c$.

Steps: Using the Concavity Test

1. Find $f'(x)$.
2. Find $f''(x)$.
3. Solve $f''(x) = 0$ and, if necessary, see where $f''(x)$ is undefined within the domain of $f(x)$ to obtain the possible points of inflection.
4. Create a number line and mark the possible points of inflection on the number line, as shown below:



5. Select a test number from each section of the number line.
6. Evaluate $f''(x)$ at each test number and record whether the sign of $f''(x)$ is positive or negative at each test point.
7. Analyze the sign changes to determine which points are points of inflection according to the Concavity Test.

Try It: Determine where each function has a point of inflection	
$f(x) = x^3 + 3x^2 - 45x + 6$	$f(x) = \frac{3}{4}(x^2 - 1)^{\frac{4}{3}}$

(TO INSTRUCTIONAL DESIGN: CREATE SOLUTIONS LINK HERE)

A Challenge: Curvature

On any curve, the **curvature** defines the rate at which a curve changes direction. If function $f(x)$ is twice differentiable on some open interval I, then the curvature of $f(x)$ is defined by the following function:

$$\kappa(x) = \frac{|f''(x)|}{(1 + [f'(x)]^2)^{\frac{3}{2}}}$$

Try It: For each function given, write a curvature function and determine the value of the curvature at $x = 3$	
$f(x) = -5x + 3$	$f(x) = \frac{2}{3}x^2$

(TO INSTRUCTIONAL DESIGN: CREATE SOLUTIONS LINK HERE)

SOLUTIONS

1. Try It for First Derivative Test
   a. Critical points are $x = -5$ and $x = 3$, where a relative maximum occurs at $x = -5$ and a relative minimum occurs at $x = 3$.
   b. Critical points are $x = 0$ and $x = \pm 1$, where a relative maximum occurs at $x = 0$ and relative minima occur at $x = \pm 1$.

2. Try It for Second Derivative Test
   a. Critical points are $x = -2$, $x = 0$, and $x = 3$, where a relative maximum occurs at $x = 0$ and a relative minimum occurs at $x = 3$.
   b. Critical points are $x = \pm 1$, where a relative maximum occurs at $x = 1$ and a relative minimum occur at $x = -1$.

3. Try It for Concavity Test
   a. A point of inflection occurs at $x = -1$.
   b. Points of inflection occur at $x \approx \pm 0.7746$.

4. Try It for Curvature
   a. $\kappa(x) = \kappa(3) = 0$.
   b. $\kappa(x) = \frac{4}{3\left(1+\frac{16}{9}x^2\right)^{\frac{3}{2}}}$ with $\kappa(3) \approx 0.019$.

Sketching the Graph of a Function

Steps to the Curve Sketching Process:
1. Obtain all possible information from $f(x)$: a. Intercepts b. End behavior c. Points of discontinuity 2. Obtain all possible information from $f'(x)$: a. Location of relative extrema. b. Intervals of increase and intervals of decrease. 3. Obtain all possible information from $f''(x)$: a. Location of points of inflection. b. Intervals of concavity. 4. Organize information: a. Take the x-values from Step 2 and Step 3 and substitute them into $f(x)$ to obtain their y-values. b. Plot the information gathered. c. Determine whether more visual information is needed. If so, select some x-values and substitute them into $f(x)$ to obtain their y-values, then plot those points. 5. Create the final graph.

Try It: Sketch graphs for the following polynomial functions.	
$f(x) = x^3 + 3x^2 - 45x + 6$	$f(x) = 2x^2 - 5x + 2$

(INSTRUCTIONAL DESIGN: INSERT SOLUTIONS LINK HERE)

When it comes to rational functions, a vertical asymptote can behave as if it were a critical point.

- Take for example $f(x) = \frac{1}{x^2}$, which has a vertical asymptote at $x = 0$.
- For $x < 0$, $f(x)$ is increasing.
- For $x > 0$, $f(x)$ is decreasing.
- Even though $f(0)$ is undefined, the behavior of the graph at $x = 0$ is similar to that of a critical point.

Similarly, a vertical asymptote can behave as if it were a point of inflection.

- Take for example $f(x) = \frac{1}{x}$, which has a vertical asymptote at $x = 0$.
- For $x < 0$, $f(x)$ is concave down.
- For $x > 0$, $f(x)$ is concave up.
- Even though $f(0)$ is undefined, the behavior of the graph at $x = 0$ is similar to that of a point of inflection.

When performing the First Derivative Test on a rational function and when performing the Concavity Test on a rational function, you must take the vertical asymptotes into account to properly obtain the increase/decrease intervals and the concavity intervals!

Try It: Sketch graphs for the following rational functions.	
$f(x) = \dfrac{x-2}{x^2 - 2x - 24}$	$f(x) = \dfrac{x^3 + 2x - 1}{x^2 - 9}$

(INSTRUCTIONAL DESIGN: INSERT SOLUTIONS LINK HERE)

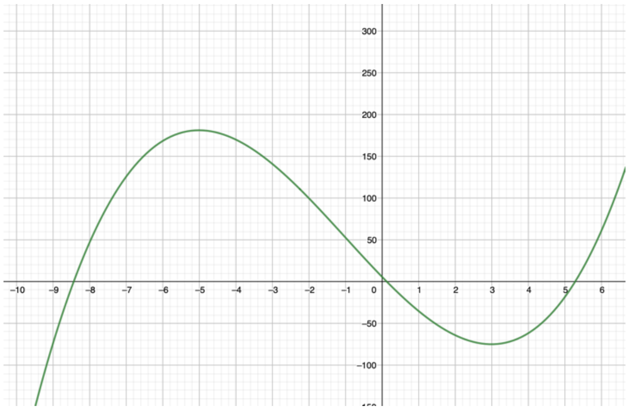
Try It: Sketch graphs for the following rational functions.	
$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$	$f(x) = \frac{3}{4}(x^2 - 1)^{\frac{4}{3}}$

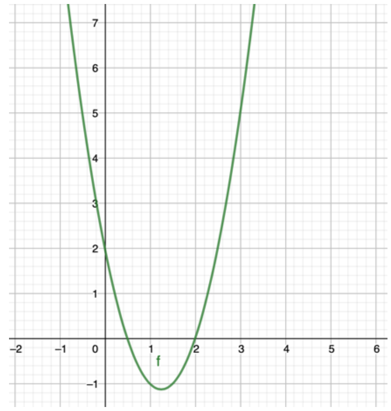
(INSTRUCTIONAL DESIGN: INSERT SOLUTIONS LINK HERE)

Solutions to Try It Problems

Try It Polynomial Functions

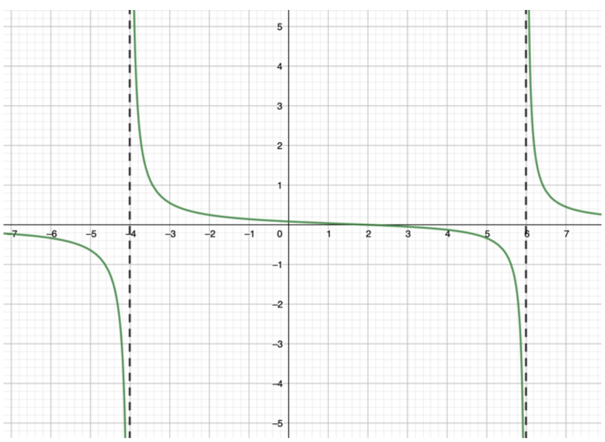
$f(x) = x^3 + 3x^2 - 45x + 6$	$f(x) = 2x^2 - 5x + 2$

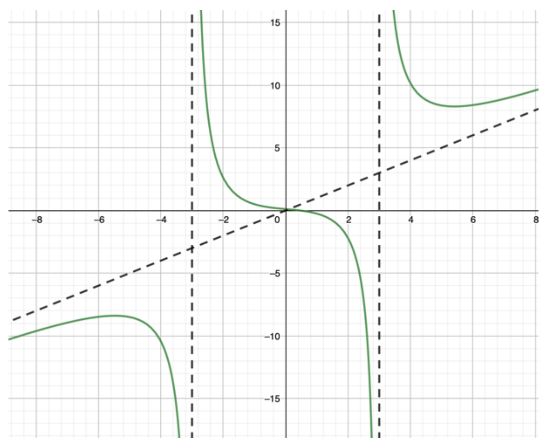




Try It Rational Functions

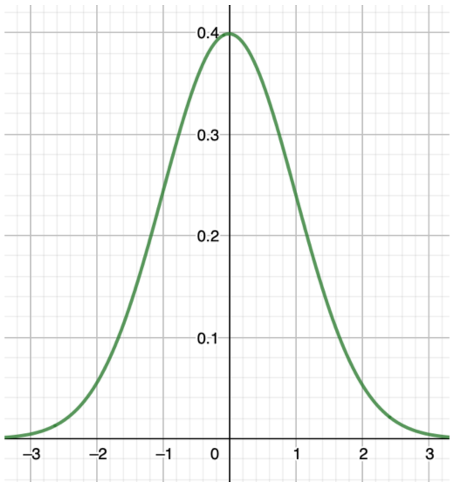
$f(x) = \dfrac{x-2}{x^2 - 2x - 24}$	$f(x) = \dfrac{x^3 + 2x - 1}{x^2 - 9}$

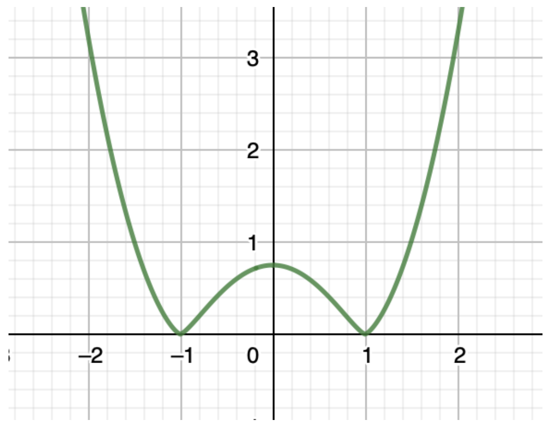




Try It Other Functions

$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$	$f(x) = \frac{3}{4}(x^2 - 1)^{\frac{4}{3}}$





Optimization

Learning Objectives:

- Understand the concept of optimization.
- Use optimization in an application problem.

One common application of calculating the minimum or maximum value is the optimize a function.

Problem 1

1. A manufacturer wants to design an open box having a square base and a surface area of 108 square inches. What dimensions will produce a box with maximum volume?
   a. What are the quantities we are trying to determine?

   b. Write the function that we are trying to maximize.

   c. Since there are 2 variables, we need to come up with an equation to solve one in term of the other. Use the fact that the surface area is 108 square inches to come up with an equation to relate the variables.

   d. Solve for the variable that is possible to solve for and substitute it into the function from part b. Find the domain that make sense for the problem. Maximize the function.

   e. Answer the question that was asked.

Strategy
1. Introduce all variables. If applicable, draw a figure and label all variables. 2. Determine which quantity is to be maximized or minimized, and for what range of values of the other variables (if this can be determined at this time). 3. Write a formula for the quantity to be maximized or minimized in terms of the variables. This formula may involve more than one variable. 4. Write any equations relating the independent variables in the formula from step 3.3. Use these equations to write the quantity to be maximized or minimized as a function of one variable. 5. Identify the domain of consideration for the function in step 4 based on the physical problem to be solved. 6. Locate the maximum or minimum value of the function from step 4. This step typically involves looking for critical points and evaluating a function at endpoints.

Problem 2

2. A rectangular garden is to be enclosed by 100 feet of fencing. One side of the garden is to be along the side of a house and therefore requires no fencing. How should the dimensions of the garden be chosen in order to maximize the enclosed area?

Practice Problems

1) A rectangular enclosure is to be constructed with an area of 600 square feet. Three sides are to be made of wood at a cost of \$7 per foot. The fourth side is to be made of cement with a cost of \$14 per foot. What dimensions minimize the cost? What is the minimum cost?

2) Find all points on the curve $y = \sqrt{x},\ 0 \le x \le 4$, that are closest to and farthest away from the point (2,0).

3) Show that the rectangle of maximal area that can be inscribed in a circle of radius 1 is a square.

4) A store can sell 20 bicycles per week at $400 each. For each $5 price reduction, they can sell 1 more bike. The bikes cost the store $200 each. Find the maximum profit. How many bikes must the store sell each week to achieve the maximal profit?

Answers to problems:

1a) The dimensions of the box, the volume of the box

1b) $V = l \cdot w \cdot h = b^2 \cdot h$ since it is a square base

1c) $108 = b^2 + 4bh$, the bottom and the 4 sides

1d) maximize when b = 6

1e) 6 in x 6 in x 3 in

2) 50 ft for the side opposite from the house, 25 ft for the other two

Answer to practice problems:

1) 20 ft for the side with cement, 30 ft for the other two

2) $\left(\frac{3}{2}, \sqrt{\frac{3}{2}}\right)$ is the closest, (4, 2) is the farthest

3) Use the equation of a circle $x^2 + y^2 = 1$ and the area of the rectangle in the circle $A = 4xy$ to show it

4) 30 bikes for a maximal profit of $8,500

L'Hospital's Rule

Indeterminate Forms of Expressions

Recall that there are forms of expressions that are ***indeterminate***. Such forms include:

$$\infty / \infty, \ (0)(\pm\infty), \ 1^{\infty}, \ 0^0, \ \infty - \infty, \ \infty^0$$

L'Hospital's Rule allows us to work with limits involving these indeterminate forms, when factoring or other algebra tricks do not help to make the expressions determinate.

Suppose that we have one of the following cases:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \ or \ \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$$

Then, we can take the derivative of the function in the numerator and of the function in the denominator, and then attempt to take the limit again. If it works, we have some limit, *L*. If not, we can take the limit a second time (or a third, etc.) and see if that gives a determinate form that we can evaluate.

Try It: Use L'Hospital's Rule to find the following limits:

$$\lim_{x \to 0} \frac{sin(x)}{x}$$

$$\lim_{x \to 2} \frac{3x^4 - 4x^2 - 8x - 16}{x^2 - 4}$$

$$\lim_{t \to \infty} \frac{e^t}{t^5}$$

Note that if an expression is in a multiplicative form (i.e. f(x)•g(x)), you can always rewrite it as f(x) / (1/g(x)), so that it is in the proper rational form.

Try It: Rewrite the limit, and then use L'Hospital's Rule to evaluate the limit

$$\lim_{x \to 0} xln(x)$$

Newton's Method

Steps to Solving a Newton's Method Problem:
1. Declare an initial guess, x_0, to the root of $f(x)$. 2. Calculate the next value in-sequence by using the formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ 3. Repeat this calculation process until you obtain the same value of x in two consecutive iterations.

When to use Newton's Method:

- To estimate the root of a function $f(x)$.
- To estimate the x-value of the intersection of two functions, which we can define by $f(x) - g(x) = 0$.
- To estimate the critical value of a function, which we can define by $f'(x) = 0$.

Try It
Estimate the root of $f(x) = x^3 - 3x + 5$ using an initial guess of $x = -3$.

(INSTRUCTIONAL DESIGN: PUT SOLUTIONS LINK HERE)

Try It
Estimate the solution to the equation $e^x = 5 - 2x$ using an initial guess of $x = 1$.

(INSTRUCTIONAL DESIGN: PUT SOLUTIONS LINK HERE)

Try It
Estimate one of the relative minimum values of $f(x) = x^2 \sin x$ using an initial guess of $x = 6$.

(INSTRUCTIONAL DESIGN: PUT SOLUTIONS LINK HERE)

Solutions to Try It Problems

1. $x = -2.27901878616659357949 1$
2. $x = 1.05869566347416083732$
3. $x = 5.08698509410227044056 3$

Antiderivatives

Think of a ladder. Each rung will either move us up the ladder, or down it. Derivatives are the steps down the ladder. Antiderivatives are the steps up the ladder. In this ladder as a function, we are moving vertically to change what the function is.

The process of finding a function from a known derivative is called the antiderivative.The process of finding the antiderivative is called antidifferentiation

Point: Though finding antiderivatives is more complicated then finding the derivative, it is often useful to ask ourselves "Whose derivative would this be?" and "What steps and methods of deriving would lead me here?". Whatever we answer with gives us insight in how to find the antiderivative.

Consider the function $f(x) = x$. Lets assume this function is the derivative of another function. Our goal is to find the original function it came from. Since our function $f(x) = x$ is a power of x, we know that to get it, we brought down the power, and then decreased the power of x by 1. To find the original function, we need to undo these steps. That means we should increase the power by 1, and then divide by the power, giving us $\frac{x^2}{2}$.

Say instead we have the function $\sin(x)$. We know that $\sin(x)$ and $\cos(x)$ are derivatives of one another, albeit with some sign changes. We know if we derive $\cos(x)$ we will get $-\sin(x)$, so the antiderivative of our function must be $-\cos(x)$.

Both of our examples so far are missing a key, important piece though. If we derive

$$f(x) = \frac{x^2}{2}$$

we will indeed get $f'(x) = x$. However, if we derive

$$f(x) = \frac{x^2}{2} + 1$$

we still get $f'(x) = x$. In fact, if we change the constant to be any other value, we will get the same derivative, since the derivative of a constant is zero.

Let $f(x)$ be a function and $A(x)$ is any antiderivative of $f(x)$. All other antiderivatives of $f(x)$ are of the form $F(x) = A(x) + C$, where C is any constant.

Now to look back at our examples more formally:

Example

1. Find the antiderivative of $f(x) = x$
   Solution:
   Since we know that an antiderivative of $f(x) = x$ is
   $$A(x) = \frac{x^2}{2}$$
   the antiderivative is written as
   $$F(x) = \frac{x^2}{2} + C$$
2. Find the antiderivative of $f(x) = \sin(x)$
   Solution:
   Since we know that an antiderivative of $f(x) = x$ is
   $$A(x) = \cos(x)$$
   the antiderivative is written as
   $$F(x) = \cos(x) + C$$

Point: The most common notation is upper case for antiderivatives. So if the function is $f(x)$ the antiderivative is $F(x)$. We often use $A(x)$ when we want to illustrate that the antiderivative is the area under the graph.

Practice

1. Find the antiderivative of the function:

    (a) $f(x) = 5$

    (b) $f(x) = 3x^2$

    (c) $f(x) = \sec^2(x)$

    (d) $f(x) = \sqrt{x} + 1$

CHAPTER 4: INTEGRATION

The last chapter introduced the idea of the antiderivative. In this chapter, we will expand on the idea of the antiderivative and the general process of integration as a preparation for Calculus II.

Learning Objectives:

1. Approximate area under the curve using a Riemann sum.
2. Evaluate the precise area under a curve using the definite integral either geometrically, as the limit of a Riemann sum, or using the Fundamental Theorem of Calculus.
3. Apply the Fundamental Theorem of Calculus to differentiate integral-defined functions.
4. Use the net-change theorem to solve future-value problems.
5. Use symmetry present in functions to evaluate a definite integral.
6. Find antiderivatives of composite functions and evaluate definite integrals of composite functions by using variable substitution.
7. Solve application problems involving definite and indefinite integrals in physics, business, and the other sciences.

Approximating Areas Under a Curve

Learning Objectives:

- Understand and know how to evaluate the sigma notation.
- Be able to use the summation operation's basic properties and formulas.
- Know how to denote the approximate area under a curve using Riemann sum and be able to find the exact area using a limit of the approximation.

Definition: Sigma Notation

$$\sum_{i=k}^{n} a_i = a_k + a_{k+1} + \cdots + a_{n-1} + a_n$$

Problems 1- 2

1. Evaluate the sum

$$\sum_{i=1}^{3} 2^i$$

2. Rewrite the expression using sigma notation and evaluate the sum.
$1 + 4 + 9 + 16 + 25$

When the number of terms is large, we can use the following rules to help simplify the problem:

$$\sum_{i=1}^{n} c = cn, \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

Problems 3 – 4

3. Evaluate the sum

$$\sum_{i=1}^{50} (i^2 + 2)$$

4. Evaluate the sum

$$\sum_{i=1}^{40} (2i^3 - i)$$

Problems 5 – 6

5. Evaluate the sum. Noted that it doesn’t start at $i = 1$ so you can’t use the rules right away (hint: add and subtract the missing terms.)

$$\sum_{i=4}^{30} (i - 3)$$

6. Express the given summation as an equation of n. Such equation is said to be in closed form.

$$\sum_{i=1}^{n} \frac{i}{n}$$

Steps for Approximating Area Under a Curve on $[a, b]$
Given a continuous, nonnegative function $f(x)$ on $[a, b]$, to approximate the area using n equal-width rectangle:
1. Compute the width of the rectangles: $\Delta x = \frac{b-a}{n}$
2. Compute the length of the rectangles, $f(x_i^*)$, depending on the point used in the subinterval: Left endpoint: $f(a + (i-1)\Delta x)$; Right endpoint: $f(a + i\Delta x)$; Midpoint: $f\left(a + \left(i - \frac{1}{2}\right)\Delta x\right)$.
3. Put what you found in step 1 and 2 into $\sum_{i=1}^{n} f(x_i^*)\,\Delta x$, and simplify.
4. If n is large, use the rules to evaluate the sum.

Problems 7 – 8

7. Approximate the area under the curve of $f(x) = x^2$ on $[1, 4]$ with 3 rectangles of equal width and using left endpoint to find the length.

a) Compute the width

b) Compute the length

c) Write out the sum and simplify.

d) Evaluate the sum.

8. Approximate the area under the curve of $f(x) = x^2$ on $[0, 3]$ with n rectangles of equal width and using right endpoint to find the length.

a) n = 24, b) n = 48, c) n is an arbitrary positive integer

Definition: Riemann sum
Let $f(x)$ be defined on a closed interval $[a, b]$, Δx_i be the width of subinterval $[x_{i-1}, x_i]$, and x_i^* be any point in $[x_{i-1}, x_i]$. A Riemann sum is defined for $f(x)$ as $$\sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Using the left endpoint, right endpoint, or midpoint to approximate the area are all examples of Riemann sum. It turns out that as the number of subintervals increase, all Riemann sums approaches the same value, regardless of the choice for x_i^*. This leads to the definition below.

Definition: Area Under a Curve
Let $f(x)$ be a continuous, nonnegative function on an interval $[a, b]$, and let $\sum_{i=1}^{n} f(x_i^*)\Delta x$ be a Riemann sum for $f(x)$. The area under the curve $y = f(x)$ on [a,b] is given by $$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Problems 9 – 10

9. Using the answer for 8c, determine the area under the curve $f(x) = x^2$ on $[0, 3]$.

10. Using left endpoint and rectangle of equal width, determine the area under the curve $f(x) = x^2$ on $[0, 3]$. Confirm that it is the same as Problem 9.

Practice Problems

Approximating Area Under the Curve and Riemann Sum

1. Evaluate the following sums.

$a. \sum_{i=1}^{3} i^2$

$b. \sum_{i=2}^{7} 3i$

$c. \sum_{i=0}^{3} \cos\left(\frac{\pi}{4} i\right)$

$d. \sum_{i=-2}^{2} (-1)^i$

2. Use the summation rules to evaluate the given sums.

$$\sum_{i=1}^{n} c = cn, \sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

$a. \sum_{i=1}^{50} (2i-4)$

$b. \sum_{i=1}^{30} i(i+1)$

$c. \sum_{i=2}^{70} (i+5)$

3. Approximate the area under the curve $f(x) = 3x + 2$ on $[1, 4]$ using 3 equal width rectangles, and using
   a. Left endpoint

   b. Right endpoint

   c. midpoint

4. Approximate the area under the curve $f(x) = \ln x$ on $[e, 5e]$ using 4 equal width rectangles, and using
   a. Left endpoint

   b. Right endpoint

5. Determine the area under the curve $f(x) = 2x + 1$ on $[1, 5]$ using
   a. Left endpoint

   b. Right endpoint

6. Determine the area under the curve $f(x) = x^2 + 2$ on $[0, 3]$ using
   a. Left endpoint

   b. Right endpoint

7. For Riemann sum, it isn't necessary for the rectangles to have the same width. In fact, a more general definition of Riemann sum is the following:

$$\lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

In this problem, we will go through a situation where it is better to use non-equal width rectangles. Let $f(x) = \sqrt{x}$ on $[0, 1]$.

a. Divide the interval into the following:

$$\left[0, \left(\frac{1}{n}\right)^2\right], \left[\left(\frac{1}{n}\right)^2, \left(\frac{2}{n}\right)^2\right], \left[\left(\frac{2}{n}\right)^2, \left(\frac{3}{n}\right)^2\right], \dots, \left[\left(\frac{i-1}{n}\right)^2, \left(\frac{i}{n}\right)^2\right], \dots, \left[\left(\frac{n-1}{n}\right)^2, 1\right].$$

What is the right endpoint for the i-th subinterval?

b. What is the length, Δx_i, of the i-th subinterval?

c. Use part a and b to find the area. Note that $\Delta x_i \to 0$ as $n \to \infty$, so you can evaluate the limit

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

Answers to problems:

1) 14

2) $\sum_{i=1}^{5} i^2$

3) 43,025

4) 1,343,980

5) 378

6) $\frac{n+1}{2}$

7a) 1 7b) $1^2, 2^2, 3^2$ 7c) $1(1) + 4(1) + 9(1)$ 7d) 14

8a) 1225/128 8b) 4753/512 8c) $\frac{18n^2+27n+9}{2n^2}$

9) 9

10) 9

Answers to Practice Problems:

1a) 14 1b) 81 1c) 1 1d) 1

2a) 2350 2b) 9920 2c) 2829

3a) 24 3b) 33 3c) 57/2

4a) 10 4b) 14

5a) 28 5b) 28

6a) 15 6b) 15

7a) $\left(\frac{i}{n}\right)^2$ 7b) $\frac{2i-1}{n^2}$ 7c) 2/3

Definite Integral

Learning Objectives:

- Be able to evaluate the definite integral of a function over a given interval using geometry.
- Be familiar with the interpretation of the definite integral of a function over an interval as the net signed area between the graph of the function and the x-axis.
- Know how to use properties of the definite integral to evaluate scalar multiples, sums, and differences of integrable functions.
- Know how to compute the average value of a function in a given interval.

We now generalize the idea of area to functions that aren't nonnegative.

Definition: Definite Integral
If $f(x)$ is a function defined on an interval $[a, b]$, the definite integral of f from a to b is given by $$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x,$$ provided the limit exists. If this limit exists, the function $f(x)$ is an integrable function on $[a, b]$.

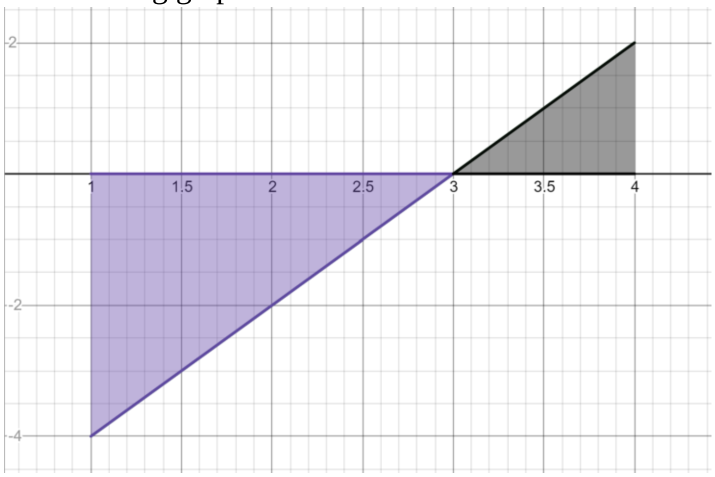
Problem 1

1. Use the definition of the definite integral to evaluate

$$\int_1^4 (2x - 6)dx$$

using right endpoint for the Riemann sum.

If done correctly, the answer to the definite integral is -3. To understand what this number represents, consider the following graph:

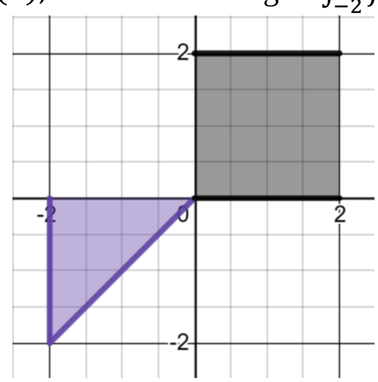


Using $A = \frac{1}{2}bh$, we have 4 for the area of the red triangle, and 1 for the area of the blue triangle. As it turns out, the Riemann sum finds the difference between them, the area above the x-axis, subtract the area below the x-axis. This difference is called the **net signed area**.

Definition: Net Signed Area
Let $f(x)$ be an integrable function defined on an interval $[a, b]$. If A_1 represent the area between $f(x)$ and the x-axis that lies above the axis, and A_2 be the area between $f(x)$ and the x-axis that lies below the axis, then the **net signed area** between $f(x)$ and the x-axis is given by $$\int_a^b f(x)\,dx = A_1 - A_2.$$

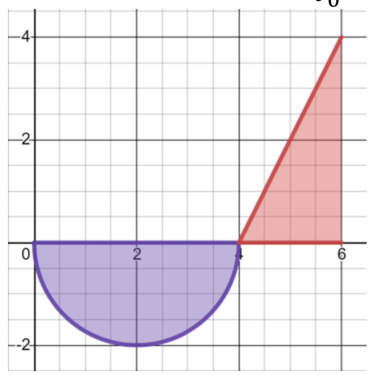
Problems 2 – 4

2. Given the following graph of $f(x)$, evaluate the integral $\int_{-2}^{2} f(x)\,dx$.



a) Where does the function change shape?
b) What are the two shapes this function created? Find the area of them using the correct area formulas.
c) Find the net signed area. This is the answer to the integral.

3. Given the following graph of $f(x)$, evaluate the integral $\int_0^6 f(x)\,dx$.



4. Evaluate the integral

$$\int_2^6 \sqrt{4-(x-4)^2}\,dx.$$

   a. Let $y = \sqrt{4-(x-4)^2}$. What geometric shape is the graph of the equation?

   b. Use the corresponding area formula to evaluate the integral.

To fine the **total area** between $f(x)$ and the x-axis on $[a, b]$, we add A_1 and A_2 together. The definite integral that represents this is given by

$$\int_a^b |f(x)| dx.$$

Problems 5 – 6

5. Find the total area for the function given in problem 2, on the interval $[-2, 2]$.

6. Find the total area for the function given in problem 3, on the interval $[0, 6]$.

Properties of the Definite Integral

Properties of the Definite Integral
Let $f(x)$ and $g(x)$ be integrable functions on the given interval.
1. $\int_a^a f(x)dx = 0$
2. $\int_a^b f(x)\,dx = -\int_b^a f(x)dx$
3. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)dx \pm \int_a^b g(x)\,dx$
4. $\int_a^b cf(x)dx = c\int_a^b f(x)\,dx$
5. $\int_a^b f(x)dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Problems 7 – 8

7. If

$$\int_a^b f(x)\,dx = 3 \text{ and } \int_a^b g(x)\,dx = 2,$$

find the value of

$a.$ $\int_a^b [2f(x) - 3g(x)]\,dx$

$b.$ $\int_b^a f(x)\,dx$

$c.$ $\int_b^b [5f(x) + 7g(x)]\,dx$

8. If

$$\int_1^{10} f(x)\,dx = 6 \text{ and } \int_1^7 f(x)\,dx = 3,$$

find the value of

$$\int_7^{10} f(x)\,dx.$$

Average Value of a Function

Definition: Average Value of a Function
Let $f(x)$ be continuous over the interval $[a, b]$. The average value of the function f_{ave} on $[a, b]$ is given by $$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Problems 9 – 11

9. Find the average value of $f(x)$ given in problem 2, over the interval $[-2, 2]$.

10. Find the average value of $f(x)$ given in problem 3, over the interval $[0, 6]$.

11. Find the average value of $f(x) = 8 - 3x$ over the interval $[0, 2]$.

Practice Problems

Definite Integral

1. Evaluate the integral $\int_0^3 (x-1)\,dx$ using the definition.

2. Sketch the region for which the given integral represents the net signed area. Evaluate the integral using an appropriate formula from geometry.

$a.\ \int_0^6 (x+2)\,dx$

$b.\ \int_{-3}^3 2x\,dx$

$c. \int_{-2}^{3} 5\, dx$

$d. \int_{-1}^{4} |x-2|\, dx$

$e. \int_{-4}^{4} \sqrt{16-x^2}\, dx$

$f. \int_{0}^{3} \sqrt{6x-x^2}\, dx,$ (hint: complete the square.)

3. Suppose that $\int_{-1}^{5} f(x)\,dx = 5$ and $\int_{-1}^{5} g(x)\,dx = -2$, compute $\int_{-1}^{5}\big(2f(x) - 3g(x)\big)\,dx$.

4. Suppose that $\int_{-2}^{8} f(x)\,dx = 7$ and $\int_{-2}^{5} f(x)\,dx = 10$, compute $\int_{5}^{8} f(x)\,dx$.

5. Suppose that $\int_{-3}^{6} f(x)\,dx = 2$ and $\int_{1}^{6} f(x)\,dx = 5$, compute $\int_{1}^{-3} f(x)\,dx$.

6. Find the average value f_{ave} for the functions from problem 2, on the given interval.

$a.\ f(x) = x + 2, \qquad [0, 6]$

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$e.\ f(x) = \sqrt{16 - x^2}, \qquad [-4, 4]$

$f.\ f(x) = \sqrt{6x - x^2}, \qquad [0, 3]$

Answers to problems:

1) -3

2a) x = 0 2b) right triangle and square, A = 2 and A = 4, respectively 2c) 2

3) $4 - 2\pi$

4a) semicircle centered at $(4,0)$, with r = 2 4b) 2π

5) 6

6) $4 + 2\pi$

7a) 0 7b) -3 7c) 0

8) 3

9) ½

10) $\frac{2-\pi}{3}$

11) 5

Answers to practice problems

1) 3/2

2a) 30 2b) 0 2c) 25 2d) 13/2 2e) 8π 2f) $\frac{9\pi}{4}$

3) 16

4) -3

5) 3

6a) 5 6b) 0 6c) 5 6d) 13/10 6e) π 6f) $\frac{3\pi}{4}$

Definite Integral

We now generalize the idea of area to functions that aren't nonnegative.

Definition: Definite Integral
If $f(x)$ is a function defined on an interval $[a, b]$, the definite integral of f from a to b is given by $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$ provided the limit exists. If this limit exists, the function $f(x)$ is an integrable function on $[a, b]$.

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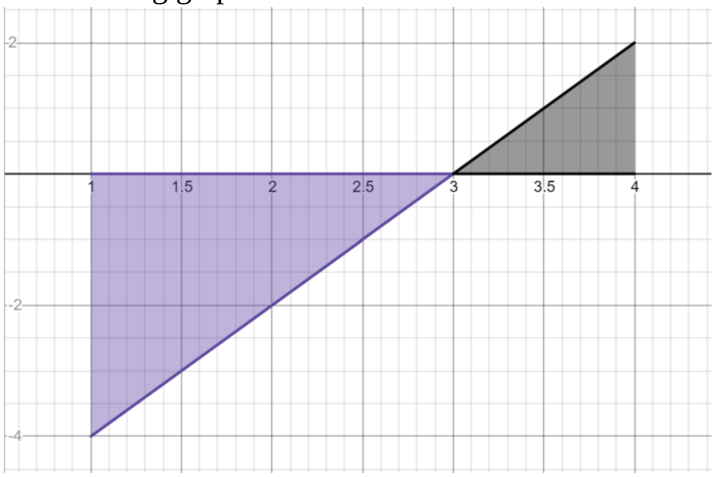

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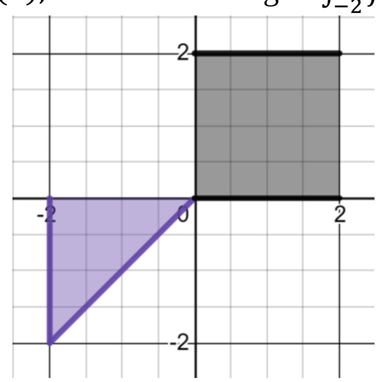

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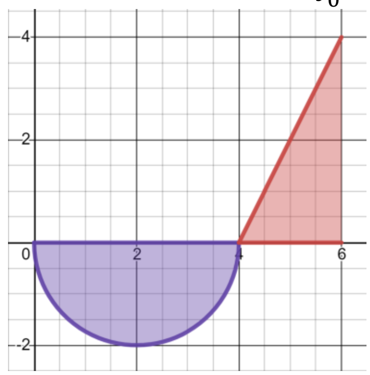

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8. If

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3. Suppose that $\int_{-1}^{5} f(x)\,dx = 5$ and $\int_{-1}^{5} g(x)\,dx = -2$, compute $\int_{-1}^{5} \big(2f(x) - 3g(x)\big)\,dx$.

4. Suppose that $\int_{-2}^{8} f(x)\,dx = 7$ and $\int_{-2}^{5} f(x)\,dx = 10$, compute $\int_{5}^{8} f(x)\,dx$.

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$f.\ f(x) = \sqrt{6x - x^2}, \qquad [0, 3]$

Fundamental Theorem of Calculus Part 1 (FTC 1)

What is the FTC?

The Fundamental Theorem of Calculus shows that differentiation and integration are inverse operations of each other.

It states that if *f* is a continuous function on the closed interval [a , b], then the function *g* defined by: $g(x) = \int_a^x f(t)dt\,, a \leq x \leq b$

is an antiderivative of *f*, that is,

$$g'(x) = f(x) \text{ or } \frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x).$$

Now, if *f* is some positive function, then we can interpret *g* as the area under the curve of *f*.

Try It: Compute $\frac{d}{dx}\int_{-5}^{x} \ln(t^2+1)\, dt$

.

Try It: Compute $\frac{d}{dx}\int_{x}^{10789} cos^5(s)ds$

Fundamental Theorem of Calculus Part 2 (FTC 2)

As the first part of the FTC relates the integral and the derivative, the second part relates the definite integral to the net change of the antiderivative.

It states that if *f* is a continuous function on the closed interval [*a* , *b*], and *F'(x) = f(x)*, then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

Or, it can be clearly written as:

$$\int_a^b g'(x)dx = g(b) - g(a)$$

This very strong theorem allows us to calculate definite integrals in two primary steps:

1. Find the antiderivative
2. Evaluate that antiderivative at the limits of integration and subtract.

Try It: Compute $\int_1^3 5x^3\, dx$

Try It: Compute $\int_0^{\pi} \sin(4x)\, dx$

The Net Change Theorem
Average Value of a Function
The Mean-Value Theorem for Integrals
Integration with Symmetry

Each of these topics is a use of integrals. We will consider them separately then as a combination.

The net change theorem states that the the final position of a value is the initial value plus the change.

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

This is identical to the result of the first part of the fundamental theorem of calculus. It states how to evaluate an integral. It also shows that an integral is not the area under a curve, but the net signed area under that curve; the result we get when we consider what is above our x-axis and what is below our x-axis.

Example

Find the net change of the function over the indicated interval:

$$y = 3x^2 + 2x - 1 \text{ over } [0, 2]$$

Solution:
This problem requires us to integrate the function over the indicated interval and evaluate.

$$\int_0^2 3x^2 + 2x - 1\,dx = x^3 + x^2 - x\Big|_0^2 = 2^3 + 2^2 - 2 - [0] = 10$$

When we substitute zero into the polynomial it becomes zero so we have nothing to subtract from the upper limit

To find the average value of a function, consider how we get an average of discrete points. We sum the points and divide by how many there are. The same idea holds here. To find the average of a function, we sum up the whole function and divide by how much of it there is. An integral, is the sum of a function, giving us the net signed area under it. If we divide it by the length of the interval, we get the average value of the function

For a continuous function $f(x)$ on $[a, b]$, the average value of the function is

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

Example

Find the net change of the function over the indicated interval:

$$y = \frac{1}{\sqrt{x}} \text{ over } [1, 9]$$

Solution:
To solve this problem, we use the formula for average value, which integrate the function and divides it by the length of the interval

$$\frac{1}{9-1}\int_1^9 \frac{1}{\sqrt{x}}\,dx = \frac{1}{8}\cdot 2\sqrt{x}\Big|_1^9 = \frac{1}{4}(\sqrt{9}-\sqrt{1}) = \frac{1}{2}$$

This means that the average height of the function across the interval is $\frac{1}{2}$.

Let us further the thought of the average value of a function, by considering where on the interval this function achieves its average value. Since the average value is $f(x)$, and our function is continuous, we must have an $x^* \in [a, b]$ where $f(x^*)$ is the average value. This is the Mean Value Theorem

Mean Value Theorem for Integrals: For a continuous function $f(x)$ on $[a, b]$, there exists at least one x^* such that

$$(b-a)f(x^*) = \int_a^b f(x)\, dx$$

Point: The mean value theorem for integrals and the average value of a function are the same formula, just solved for different parts. The average value of a function wants $f(x)$, where the mean value theorem wants the x where the average occurs.

Example

Find the x^* that makes the mean value theorem hold for the function over the indicated interval:

$$y = \frac{1}{\sqrt{x}} \text{ over } [1, 9]$$

Solution:
We already know, from the previous example, that

$$f(x) = \frac{1}{2}$$

So we now want to solve this equation for x. We substitute the function in for $f(x)$ and solve

$$\frac{1}{\sqrt{x}} = \frac{1}{2} \Rightarrow \sqrt{x} = 2 \Rightarrow x = 4$$

So on the interval $[1, 9]$, the average height of $\frac{1}{2}$ occurs when $x = 4$

Point: Think of the mean value theorem of integrals like leveling out the sand in a box. The sand originally has peaks, and valleys. The mean value theorem is judging where would the sand level to, making it a rectangular solid.

If a function possesses symmetry, we can use it for easing the integration. This can result in less work in the calculation, or easier values to use in the calculation. For a function $f(x)$, the symmetry most often used is across the y-axis.

Point: When looking for symmetry, the graph of the function is the most useful way to see it, as the function may not seem symmetric until we see it.

Point: Be careful with what you are integrating and if it is symmetric.

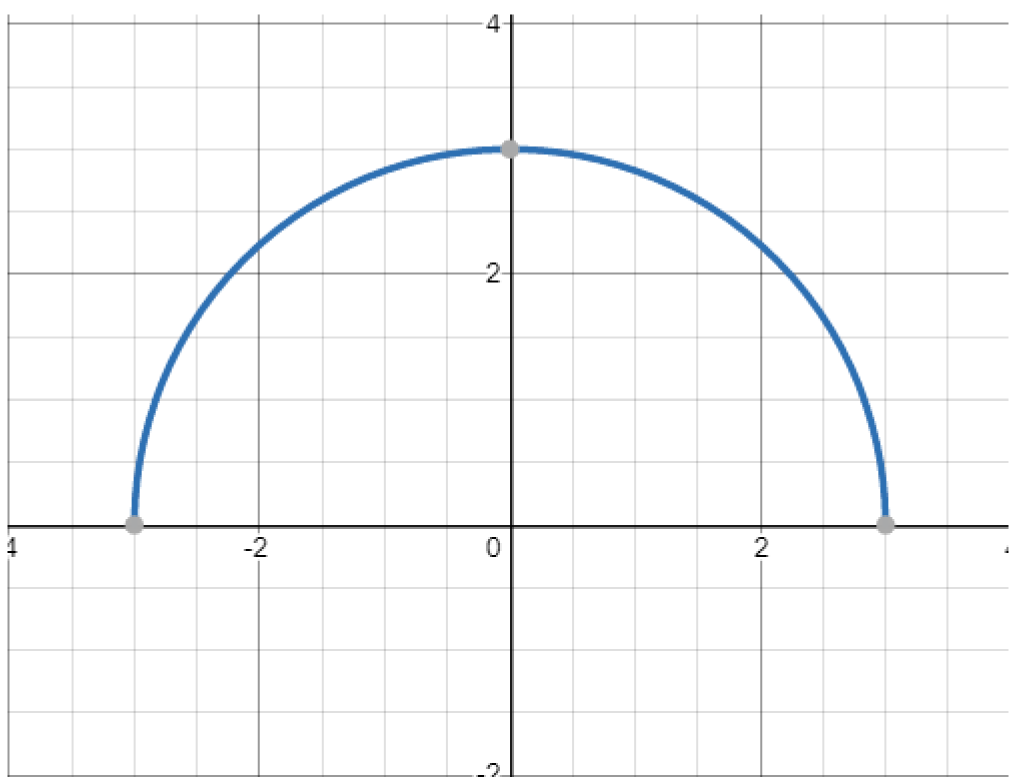
Example

$$f(x) = \sqrt{9 - x^2}$$

is the upper half of a circle of radius 3 centered at the origin. This function is symmetric about $x = 0$, so

$$\int_{-r}^{r} \sqrt{9 - x^2}\, dx = 2 \int_{0}^{r} \sqrt{9 - x^2}\, dx$$

where we only consider the right half of the semicircle. If we look at the graph we can see the symmetry.



Practice

1. For the following function $f(x) = x^2 - 2x + 3$ on $[0, 3]$

   (a) Find the average value of the function over the interval

   (b) Find the x^* that makes the mean value theorem true for the function on the interval

2. For the following function $f(x) = x\sqrt{16 - x^2}$ on $[-4, 4]$

   (a) Find the average value of the function over the interval

   (b) Find the x^* that makes the mean value theorem true for the function on the interval

Substitution with Indefinite Integrals.

Theorem: Substitution with Indefinite Integrals

Let $u = g(x)$, where $g'(x)$ is continuous over an interval, let $f(x)$ be continuous over the corresponding range of g, and let $F(x)$ be an antiderivative of $f(x)$.Then,

$$\int f\left[g(x)\right] g'(x)dx = \int f(u)du = F(u) + C = F\left(g(x)\right) + C$$

Problem-Solving Strategy: Integration by Substitution

1. Look carefully at the integrand and select an expression $g(x)$ within the integrand to set equal to u. Let's select $g(x)$, such that $g'(x)$ is also part of the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)dx$ into the integral.
3. We should now be able to evaluate the integral with respect to u. If the integral can't be evaluated we need to go back and select a different expression to use as u.
4. Evaluate the integral in terms of u.
5. Write the result in terms of x and the expression $g(x)$.

Examples

Example 1. Use substitution to find the antiderivative $\int 3x^2\left(x^3 - 3\right)^2 dx$.

The first step is to choose the expression for u. Let $u = x^3 - 3$. Then $du = 3x^2dx$. and we already have du in the integrand. Let's write the integral in terms of u.

$$\int 3x^2\left(x^3 - 3\right)^2 dx = \int \left(x^3 - 3\right)^2 3x^2\, dx = \int u^2du$$

Let's evaluate the integral in terms of u:

$$\int u^2du = \frac{u^3}{3} + C = \frac{\left(x^3 - 2\right)^3}{3} + C$$

We can always check our answer by differentiating the function we got as a result of integration. (Try it!).

Example 2. Use substitution to find $\int \cos t\sqrt{\sin t + 1}\, dt$.

We know that the derivative of $\sin t$ is $\cos t$. Let $u = \sin t + 1$. The $du = \cos t\, dt$. Substituting these into the integral we get

$$\int \cos t\sqrt{\sin t + 1}\, dt = \int \sqrt{\sin t + 1}\cos t\, dt = \int \sqrt{u}\, du$$

Evaluating the integral in terms if u, we get

$$\int \sqrt{u}\,du = \int u^{1/2}\,du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3}u^{3/2} + C$$

Putting the answer back in terms of t we get

$$\int \cos t\sqrt{\sin t + 1}\,dt = \frac{2}{3}(\sin t + 1)^{3/2} + C$$

Exercises

In the following exercises use the indicated u-substitution to find the each integral.

1. $\int (x-2)^2\,dx;\ u = x-2$
2. $\int 2x(x^2-2)^{-2}\,dx;\ u = x^2-2$
3. $\int \frac{x}{\sqrt{x^2-2}}\,dx;\ u = x^2-2$
4. $\int (x-1)(x^2-2x)^3\,dx;\ u = x^2-2x$

In the following exercises, use a suitable change of variables to determine the indefinite integral.

5. $\int t^2\left(t^3-1\right)^{10}\,dt$
6. $\int \cos^3 x \sin x\,dx$
7. $\int x\sec^2\left(x^2\right)\,dx$
8. $\int \frac{\ln x}{x}\,dx$
9. $\int \frac{\tan^2 x}{x^2+1}\,dx$

Substitution with Definite Integrals.

Theorem: Substitution with Definite Integrals

Let $u = g(x)$ and let $g'(x)$ be continuous over an interval $[a, b]$, and let f be continuous over the range of $u = g(x)$. Then,

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Examples

Example 1. Use substitution to evaluate $\int_{-1}^{0} x\left(2x^2 - 3\right)^5 dx$.

Let $u = 2x^2 - 3$, so $du = 4x\,dx$. Because the original function includes only one factor of x and $du = 4x\,dx$, we will multiply both sides of the du equation by 1/4.

$$du = 4x\,dx$$

$$\frac{1}{4}du = x\,dx.$$

To adjust the limits of integration, note then when $x = -1$, $u = 2(-1)^2 - 3 = -1$, and when $x = 0$, $u = 2(0)^2 - 3 = -3$. Then

$$\int_{-1}^{0} x\left(2x^2 - 3\right)^5 dx = \int_{-1}^{-3} u^5\,\frac{1}{4}du$$

Evaluating this expression

$$\int_{-1}^{-3} u^5\,\frac{1}{4}du = \frac{1}{4}\int_{-1}^{-3} u^5\,du = \frac{1}{4}\frac{u^6}{6}\Big|_{-1}^{-3} = \frac{1}{24}\left[(-1)^6 - (-3)^6\right] = -\frac{91}{3}$$

Example 2. Use substitution to evaluate $\int_0^1 x^2\cos\left(\frac{\pi}{2}x^3\right)\,dx$.

Let $u = \frac{\pi}{2}x^3$, then $du = \pi x^2\,dx$. Multiplying both side of the last equality by $\frac{1}{\pi}$ we get

$$du = \pi x^2\,du$$

$$\frac{1}{\pi}du = x^2\,dx$$

To change the limits of integration in agreement with our chosen substitution we note that when $x = 0$, $u = \frac{\pi}{2}\cdot 0^3 = 0$, and when $x = 1$, $u = \frac{\pi}{2}\cdot 1^3 = \frac{\pi}{2}$. Then

$$\int_0^1 x^2\cos\left(\frac{\pi}{2}x^3\right)\,dx = \int_0^1 \cos\left(\frac{\pi}{2}x^3\right)\,x^2dx = \int_0^{\pi/2}\cos(u)\cdot\frac{1}{\pi}du = \frac{1}{\pi}\int_0^{\pi/2}\cos(u)du$$

Evaluating the expression, we get

$$\frac{1}{\pi}\int_0^{\pi/2}\cos(u)du = \frac{1}{\pi}(-1)\sin(u)|_0^{\pi/2} = -\frac{1}{\pi}\left[\sin(0) - \sin(\pi/2)\right] = \frac{1}{\pi}$$

Exercises

In the following exercises, use a change of variables to evaluate the definite integral.

1. $\displaystyle\int_0^1 x\sqrt{1-x^2}\,dx$

2. $\displaystyle\int_0^1 \frac{t}{\sqrt{1+t^2}}\,dt$

3. $\displaystyle\int_0^1 \frac{x^2}{1+x^3}\,dx$

4. $\displaystyle\int_0^{\pi/4} \sec^2 x \tan x\,dx$

5. $\displaystyle\int_0^{\pi/4} \sin^2(2x)\cos(2x)\,dx$

6. $\displaystyle\int_0^3 (x+1)(x^2+2x)\,dx$

7. $\displaystyle\int_0^{\pi/6} \frac{\sin\theta}{\cos^4\theta}\,d\theta$

Integrals Involving Exponential and Logarithmic Functions

Integrals Involving Exponential Functions

$$\int e^x \, dx = e^x + C$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + C$$

Notice that the integral of $f(x) = e^x$ is simply $e^x + C$.

More generally, the integral of e^u du (where *u* is some function) is: $\frac{1}{u'}e^u + C$

Try It: Compute $\int e^{3x+4}dx$

Try It: Find the price-demand equation for a particular brand of toothpaste at a supermarket chain when the demand is 100 tubes per week at $2.75 per tube, given that the marginal-price-demand function, $p'(x)$, for x number of tubes per week is given by: $p'(x) = -0.015e^{-0.01x}$

If the supermarket chain sells 150 tubes per week, what price should it set?

Integrals Involving Logarithmic Functions

Integrating functions of the form $\frac{1}{x}$ result in the absolute value of the natural logarithm function.

Notice that this function can be written as x^{-1} and thus the power rule will not work. Instead:

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

More generally, we have $$\int \frac{u'(x)}{u(x)}\,dx = \ln|u(x)| + C$$

Try It: Find the antiderivative of $\frac{1}{x+2}$

Try It: Find the definite integral: $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+cosx}\,dx$

Integrals Resulting in Inverse Trigonometric Functions

To discuss the integrals that result in inverse trigonometric functions, let us first review the derivatives of our inverse trigonometric functions.

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2} \qquad \frac{d}{dx}\csc^{-1}(x) = -\frac{1}{|x|\,\sqrt{x^2-1}}$$

Point: There are only three derivatives for these functions, as the other set, $\cosh(x)$, $\coth(x)$, and $\operatorname{csch}(x)$, are just the negatives for the other three. This means we only need three integrals.

$$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}(x) + C \qquad \int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{1+x^2}dx = \tan^{-1}(x) + C \qquad \int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1}|x| + C \qquad \int \frac{1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$$

Point: Notice the second set of integrals just tells us how to handle the case where the constant in the integral is not a 1. If you forget this form, it can be found by factoring out a^2 from the denominator. Also, arcsine is not missing the $\frac{1}{a}$ in front, as it actually has cancelled out.

Another important note is the difference between the three. It is easiest to tell when the problem calls for arctan, due to the addition and lack of square root in the denominator. For the arcsine and arcsecant, the main difference to look for is what has the negative sign. In the arcsine formula, the x term is negative. In the arcsecant formula, the constant is negative. Yes, the arcsecant has a variable in the denominator outside the square root, but that can be manufactured in a problem or found in a substitution, etc. The main feature is what is negative inside the square root.

Example

1. Find the integral:

$$\int \frac{2}{x^2+9}dx$$

Solution:
We see that this problem is the x^2 term plus the constant, so this is arctan.

$$\text{If } a^2 = 9 \text{ then } a = 3$$

and the coefficient in the numerator just multiples to the solution

$$2\int \frac{1}{3^2+x^2}dx = \frac{2}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

2. Find the integral:

$$\int \frac{1}{\sqrt{x}\sqrt{1-x}}dx$$

Solution:
It is tempting to consider this problem similar to the arcsecant problem. The issue is arcsecant has the variable minus the constant in the radical, and this has the constant minus the variable in the radical. To see how this makes it arcsine let us consider the substitution

$$\text{If } x = u^2, \text{ then } u = \sqrt{x}, \text{ and } \frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

It can often be useful to consider what we want to see in the problem, then to find the substitution from it. We know to use any of these formulas, we need a squared variable, so we will substitute x to be u^2. This gives us

$$2\int \frac{1}{\sqrt{1-u^2}}\,du$$

Which is exactly the arcsine function. Just remember to convert back to the original variable

$$2\int \frac{1}{\sqrt{1-u^2}}\,du = 2\sin^{-1}(u) + C = 2\sin^{-1}(\sqrt{x}) + C$$

Point: The most common notation is upper case for antiderivatives. So if the function is $f(x)$ the antiderivative is $F(x)$. We often use $A(x)$ when we want to illustrate that the antiderivative is the area under the graph.

Practice

1. Calculate the following integrals

   (a) $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$

   (b) $\displaystyle\int \frac{dx}{3x\sqrt{x^2-1}}$

   (c) $\displaystyle\int \frac{dx}{x[1+\ln^2(x)]}$

Area Between Curves

Steps: Finding Area Between Two Curves with Vertical Heights

Given two functions $f(x)$ and $g(x)$ with area captured in between them:
1. If necessary, determine the interval over which the area exists by solving $f(x) = g(x)$.
2. Determine the heights of a rectangle within the region: $H(x) = g(x) - f(x)$.
    a. Use proper judgement in setting this up based on which function has greater values.
    b. This may result in multiple height functions depending on the nature of the region.
3. Integrate the height function over the appropriate interval.

Try It #1
Find the area in between the curves defined by $f(x) = x + 4$ and $g(x) = 3 - \frac{1}{2}x$ over the interval [1,4].

(TO INSTRUCTIONAL DESIGN: CREATE SOLUTIONS LINK HERE)

Try It #2
Find the area in between the curves defined by $f(x) = 6 - x$ and $g(x) = 9 - \frac{1}{4}x^2$.

(TO INSTRUCTIONAL DESIGN: CREATE SOLUTIONS LINK HERE)

Try It #3
Find the area between the curves defined by $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[0, \pi]$.

(TO INSTRUCTIONAL DESIGN: CREATE SOLUTIONS LINK HERE)

Try It #4
Find the area between the curves defined by $f(x) = x^2$, $g(x) = 2 - x$ and the x-axis.

(TO INSTRUCTIONAL DESIGN: CREATE SOLUTIONS LINK HERE)

When needed, a function can be redefined to be of the form $x = f(y)$, in which case the rectangles within the area would have horizontal heights instead of vertical heights. The process of finding area between curves does not change, only the representation of that area.

Try It #5
Repeat the problem from Try It #4 by redefining the functions to be functions of y.

(TO INSTRUCTIONAL DESIGN: CREATE SOLUTIONS LINK HERE)

SOLUTIONS

1. Try It #1: $\frac{57}{4}$
2. Try It #2: $\frac{64}{3}$
3. Try It #3: $2\sqrt{2}$
4. Try It #4: $\frac{5}{6}$
5. Try It #5: $\frac{5}{6}$