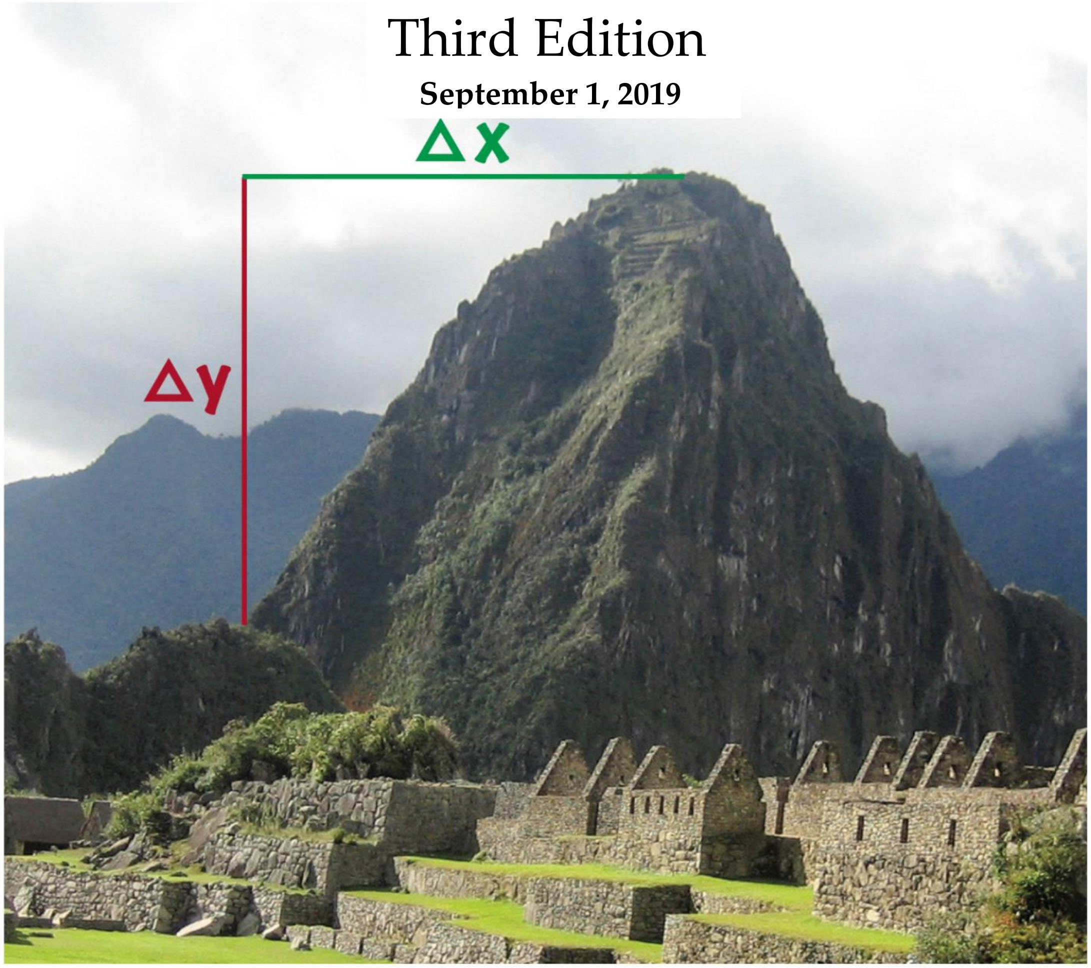
MATH 021

Introductory Algebra

Third Edition

September 1, 2019



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Preface

In 1996, the Brookdale Mathematics Department created an Introductory Algebra course for our non-STEM students. Math faculty consulted other departments with courses that had Introductory Algebra as a prerequisite, and created an algebra course that truly served many non-STEM disciplines. The Department selected a textbook that was as innovative as the course. However, it did not match our preferred order of topics, and it did not include some prerequisite topics requested by other departments, such as Probability and Statistics.

For years math faculty created support curriculum in hopes of eventually developing our own textbook. We wanted to provide non-STEM students with a high-quality and academically rigorous book that supported our course learning outcomes, for free or at minimum cost. Finally, with the advent of Open Educational Resources (OER) and the support of Brookdale Community College, we decided to build a book that combined a remix of OER textbook definitions with our own curriculum.

This text was piloted in the Spring 2017 semester by math faculty who volunteered to use the book and contribute suggestions. Since then, we received suggestions and edits from faculty who taught the course. This Third Edition reflects those suggestions and changes. We remain appreciative of the support and encouragement we received from our families, as well as Brookdale administrators, staff, and faculty.

Barbara Tozzi, Ed.D.

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September 1, 2019

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including the following Math Faculty:

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1.1 Problem Solving and Numerical Representations

Introduction

Everyone must have felt at least once in his or her life how wonderful it would be if we could easily solve a problem using one method that works every time. Problem solving can be challenging, and there are no universal approaches one can take to solving problems. Basically one must explore possible methods until recognizing the path to a solution. Guessing can help, and algebraic thinking can be involved. In general, as one gains experience in solving problems, one develops one's own techniques and strategies. Thus the guessing is not arbitrary. Rather, it is educated.
In this section we are going to learn a framework for problem solving and get a glimpse of strategies that are often used by experts. The strategies are based on the work of George Polya. For further study, read: G. Polya, How to Solve It, A New Aspect of Mathematical Method, Second Ed., Princeton University Press, Princeton, NJ, 1985.

A Framework for Problem Solving

The following four phases can be identified in the process of solving problems:

(1) Understanding the problem

(2) Making a plan of solution

(3) Carrying out the plan

(4) Looking back i.e. verifying

Before we apply this process, it is important to introduce some algebraic vocabulary.

Definitions and Key Concepts
Variable: A variable, usually represented by a letter or symbol, can be defined as: • A quantity that may change within the context of a mathematical problem. • A placeholder for a specific value.

Definitions and Key Concepts
Input-Output table: An input-output table is the numerical representation of a rule. This table has at least two columns. The first column (input) is for values that you know; the second column (output) lists the answer that you get for each input value, after applying an appropriate rule.
The rule for an input-output table is the symbolic representation. Rules are equations that begin with: $y =$. A rule can also be called a formula.

Example 1:

When the Mathematics Department moved to its current location, faculty considered various ways to place cubicles in the large, open space. Each cubicle contains a desk for one professor, and uses three partitions for sides. One suggestion for how to place the cubicles, was to make one long row. Figure 1 (below) shows this arrangement for five faculty.

How many partitions will be needed to arrange the cubicles in one long row for 30 faculty?

Figure 1.

Solution:

(1) **Understanding the problem** – This is a "find it" type of problem. If we use the cubicle layout in Figure 1 above, we need to find how many partitions will be needed for 30 cubicles – one cubicle for each professor.

(2) **Making a plan of solution** – Since a pattern seems to be involved in this problem (each new cubicle requires an additional two partitions), it may be helpful to **create an input-output table** that lists the number of cubicles, and the number of partitions that would be needed. Creating a table is known as a **numerical representation**.

(3) **Carrying out the plan –** The first cubicle needs three partitions, but the second cubicle only needs two more partitions since one partition is shared. Each new cubicle needs two additional partitions.

To read the table, read across each row individually. For example, if you only wanted to make three cubicles, then you would need 7 partitions.

Input, x **Number of Cubicles**	**Output, y** **Number of Partitions**
1	3
2	5
3	7
4	9
5	11

Following this pattern, it would be cumbersome to make a table that has 30 rows to represent 30 cubicles, so this is where algebra can be helpful. Using the variables x (to represent the number of cubicles) and y (to represent the number of partitions), the **rule** (or formula) to find the number of partitions is $y = 2x + 1$.

For now, you will be given the rule when you work with tables. Eventually, you will learn how to come up with the rule yourself. Some of you may even be able to guess the rule now!

(4) **Looking back i.e. verifying** – Let's see if the rule works: for 4 cubicles, we replace x with 4:

$$y = 2(4) + 1$$
$$y = 8 + 1$$
$$y = 9$$

Look back at Figure 1. For four cubicles, were there 9 partitions? YES! The rule works, so now we can answer the original question and apply it for 30 faculty, or $x = 30$: $y = 2(30) + 1$, $y = 61$.

Answer: 61 partitions will be needed to make cubicles in one long row for 30 faculty.

Practice Using Input-Output Tables

Example 2:

American Fitness charges a one-time registration fee of \$50, plus \$25 per month of membership at its gym. Answer the following questions using the rule $y = 25x + 50$, where x represents the number of months of membership and y represents the total cost of membership for that number of months.

a) How much will it cost a student to become a member for two years?
b) How much will it cost a student to become a member for thirty months?

Solution part a):

(1) **Understanding the problem** - If we let x represent the number of months of membership, and y represent the total cost of membership for that number of months, the rule $y = 25x + 50$ represents this situation.

(2) **Making a plan of solution** – Again a table will be helpful since there is a pattern – the same amount is being added to the cost, depending on the number of months.

(3) **Carrying out the plan** – We use the rule: $y = 25x + 50$ to complete the table below:

Input, x **Number of months of membership**	**Output, $y = 25x + 50$** **Total cost of membership**
0	50
6	200
12	350
18	500
24	650

(4) **Looking back i.e. verifying** – This is where you need to make sure that your answer "makes sense." Since you are asked for the price of a two-year membership, be careful to realize that two years is the amount paid for 24 months. If you only let $x = 2$, the answer will be the cost of membership for two months!

Answer: A two-year (24 month) membership will cost the student $650.

Solution part b):

Since we know the rule: $y = 25x + 50$ works for this situation, where x represents the number of months of membership, we can let $x = 30$.

We have: $y = 25(30) + 50$, $y = 800$.

Answer: A 30 month membership will cost the student $800.

Example 3:

Some phones can be used on a "pay as you go" system. For example, Company T charges $0.20 per minute for airtime on its prepaid cell phones. Customers pay $20.00 to "fill" their phone with airtime. The rule is $y = 20 - 0.20x$, where x represents the number of minutes and y represents the amount of money remaining on the Company T phone. Find how many minutes of airtime a customer will receive for $20.

Solution:

The rule $y = 20 - 0.20x$ gives the amount of money remaining on the phone (y) after x minutes of airtime have been used.

Complete the table below:

Input, x **Minutes of airtime that have been used**	**Output, $y = 20 - 0.20x$** **Amount of money remaining on the phone**
10	18
20	16
30	14
50	10
100	0

<u>Answer</u>: For $20, the customer will receive 100 minutes of airtime.

Conditional Rules

Some input-output tables require more than one rule, or **conditional rules**, depending on the input value. Example 4 illustrates that shipping and handling fees can depend on the amount of money you are spending. There are five different fees (output), depending on the total of purchases (input).

Example 4:

Lydia is shopping at an online website. She selects $99.85 in merchandise, and then uses the table below to determine the additional shipping and handling fees.

Input, Total of purchases	Output, Shipping and handling fee
Up to $25	$4.95
$25.01 - $50	$5.95
$50.01 - $75	$6.95
$75.01 - $100	$7.95
More than $100	Free

a) How much will she spend on shipping and handling for her $99.85 purchase?

b) If she decides to purchase one more item for $5, how much will she pay for shipping and handling?

Answers: a) Shipping and handling for a $99.85 purchase is $7.95.

b) If the total of her purchases is $104.85, then shipping and handling is free.

Practice Exercises:

For exercises 1 – 3 use the table below, which lists the USPS rate to mail a small package.

Input, Weight of Package	Output, Cost to mail
3 ounces or less	$2.45
More than 3 ounces	$2.45 plus $0.19 for each additional ounce over 3

1) How much does it cost to mail a small package that is 3 ounces?

2) How much does it cost to mail a small package that is 4 ounces?

3) How much does it cost to mail a small package that is 10 ounces?

1.1 SUMMARY

1. A ________________ is a letter or symbol use to represent an unknown quantity.

2. According to George Polya, when solving a word problem, the first step is to _____________ ____________________________________.

3. Input-output tables are ___________________ representations of an input-output rule.
(symbolic or numerical)

4. The input-output rule is the _______________________ representation of an input-output table. (symbolic or numerical)

5. Complete the input-output table below for the rule: The output is three less than the input.

In symbolic form, the rule is: $y = x - 3$

Input, x	**Output, $y = x - 3$**
–2	
–1	
0	
1	
2	

6. The minimum monthly payment on a credit card is summarized in the following table:

Input, Balance	**Output, Minimum monthly payment**
\$1750 and less	\$35
\$1750.01 and more	2% of the balance

a) The balance on a credit card is \$500. What is the minimum monthly payment?

b) The balance on a credit card is \$1750. What is the minimum monthly payment?

c) The balance on a credit card is \$2000. What is the minimum monthly payment?

1.2 Real Numbers and Translating Algebraic Expressions

Set Notation and Sets of Numbers

The following definitions will be used extensively throughout this book.

Definitions and Key Concepts	
Set notation	Sets of numbers are listed in braces { }. The ellipsis or three dots (…) is used for sets with infinitely many elements.
Real numbers	A real number is any number that is the coordinate of a point on the real number line. The set of real numbers includes rational numbers and irrational numbers. Watch this video to learn about real numbers
Natural numbers	$\{1, 2, 3, 4, \ldots\}$
Whole numbers	$\{0, 1, 2, 3, 4, \ldots\}$
Integers	$\{\ldots -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$
Rational numbers	Rational numbers are real numbers that can be written in the form $\frac{a}{b}$, where a and b are integers, and $b \neq 0$.
Irrational numbers	Irrational numbers are numbers that cannot be written as a quotient. The decimal representations of irrational numbers are non-terminating and non-repeating.
Approximate values	Since irrational numbers are non-repeating, non-terminating decimals, we use a calculator to evaluate them and round the answer. For example, use your calculator to find $\sqrt{50}$. We say that $\sqrt{50} \approx 7.07$, rounded to the nearest hundredth. Notice the use of the approximately equal to sign ($\approx$). Use this sign for approximate (or rounded) values.

Example 1:

Write the sets of numbers to which each of the following numbers belongs:

a) $-\sqrt{16}$ **b)** $\frac{1}{5}$ **c)** $\sqrt{20}$

__Answers__:

a) $-\sqrt{16} = -4$ which is real, an integer and a rational number (since it can be written as $-\frac{4}{1}$).

b) $\frac{1}{5}$ is real, and a rational number.

c) $\sqrt{20}$ is real, and an irrational number.

Practice Exercises: Sets of numbers

For exercises 1 – 5, answer true or false and explain your answer.

1) Every natural number is a whole number.

2) Every integer is a rational number.

3) π is a rational number.

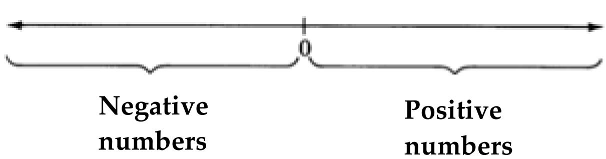
4) Every integer is a natural number.

5) Zero is an example of an integer that is a natural number.

The Real Number Line

In our study of algebra, we will use several sets of numbers. The real number line allows us to **visually** display the numbers in which we are interested.

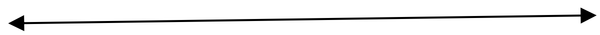
A line is composed of infinitely many points. To each point we can associate a unique number, and with each number we can associate a particular point.



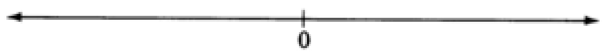
The number 0 is neither positive nor negative.

To construct the real number line:

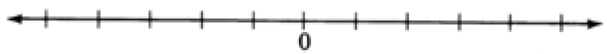
1) Draw a horizontal line.



2) Choose any point on the line and label it 0. This point is called the origin.



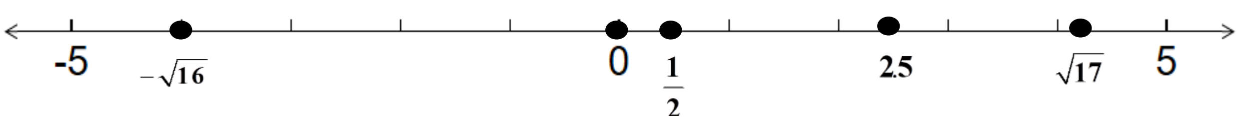
3) Choose a convenient length. This length is called "1 unit." Starting at 0, place tick marks to mark this length off in both directions, being careful to have the lengths look like they are about the same. Decide on a scale, and assign a real number value to each tick mark.



Example 2:

Construct a real number line and label the location of the given numbers: $-\sqrt{16}$, $\frac{1}{2}$, 0, $\sqrt{17}$, 2.5

__Answers__:



Practice Exercises: The real number line

For exercises 6 – 8, construct a real number line for each exercise and label the location of the given numbers.

6) 1.3, $-\frac{1}{4}$, -3, π, 4.25

7) $-2\frac{1}{8}$, 4, 0.7, $-1\frac{1}{3}$, $\sqrt{4}$

8) Draw a number line and label the even integers from – 6 to 6.

Algebraic Expressions

Definitions and Key Concepts	
Constant	A letter or symbol that represents a specific number (known or unknown) is called a constant.
Numerical coefficient	In the expression $-5x$, the number -5 is called the numerical coefficient of the quantity x. Often, the numerical coefficient is just called the coefficient. It is the number connected to a variable or variables by multiplication.
Expression	An algebraic expression is a mathematical statement that can contain numbers, variables, and operations (addition, subtraction, multiplication, division, etc).

Example 3:
Identify the numerical coefficients and constants in the following expression: $-4x^2+2y-1$.

<u>Answer</u>:
The coefficients are – 4 and 2, and the constant is – 1.

Practice Exercises: Algebraic Expressions

For exercises 9 – 12, identify the numerical coefficients and constants in the following expressions:

9) $-4xy$

10) $5x^2-x-3$

11) $2a^2+\pi$

12) $4xy-\sqrt{5}$

Translating Words into Mathematical Symbols

Mathematics Dictionary (A more extensive list is in your Canvas course)		**Examples**
Addition	Sum, sum of, added to, increased by, more than, and, plus	$-x+y$
Subtraction	Difference, minus, subtracted from, decreased by, less, less than	$x-y$
Multiplication	Product, the product of, of, multiplied by, times, per, twice or double ($2x$), half of a number $\left(\frac{1}{2}x\right)$, triple ($3x$)	$2\cdot 3;\ 2(3);\ 2xy$
Division	Quotient, divided by, ratio, per	$x\div 3;\ \frac{x}{3};\ 3\overline{)x}$
x (or another variable)	A number, an unknown quantity, an unknown, a quantity, the input	A number increased by 2: $x+2$
=	Equals, is equal to, is, the result is, becomes, the output is equal to	$3-4=-1;$ $y=3x-4$

Example 4:

Translate each phrase or sentence into a mathematical expression or rule.

a) The sum of a number and 10.

b) Sales tax is 7% of the price of an item. If the price of an item is x and the sales tax is y, write a rule representing the sales tax.

<u>**Answers**</u>**:**

a) $x+10$
b) $y=0.07x$

Practice Exercises: Translating words into mathematical symbols

For exercises 13 – 26, translate each phrase or sentence into a mathematical expression or rule.

13) Eight less than a number

14) A quantity less eight

15) One-fifth of a number

16) The difference between the input and two

17) The quotient of the input and six

18) The sum of one and twice the input

19) The product of 5 and twice the input

20) The output is half of the input.

21) The output is the difference between twice the input and 5.

22) The quotient of the input and 3, decreased by three, is the output.

23) Tuition at a local college costs $550 per credit plus fees. Fees are $300 each semester. Write an equation that represents the cost of tuition, y, for the fall semester, if a student takes x credits.

24) A tourist purchases a subway card for $25. If each train ride costs $2.50, write a rule that represents the balance remaining on the subway card, y, after taking x rides.

25) Cresto credit cards give 4% cash back on gas purchases. If a customer spends x dollars on gas this month, write a rule that represents how much she will receive in cash back, y, this month.

26) Renting a car for one week costs $250 plus $0.95 per mile. If a customer drives x miles this week, write a rule that represents how much it will cost to rent a car, y.

Translating and Building Input-Output Tables

Practice Exercises: Translating and building input-output tables

For exercises 27 – 28, write a rule that represents each situation and use the completed input-output table to answer the appropriate questions.

27) A student purchases a copy card from the local library for $5. Each copy costs $0.10.

a) Write a rule that shows how much money remains on the copy card if x copies have been made.

b) For the number of copies made, x, create an input-output table for the balance that remains on the card, y. For x, use integers from 0 to 50 in steps of 10.

Input, x **Number of copies made**	**Output, y** **Balance remaining on the copy card**

c) If 10 copies are made, what is the balance that remains on the card?

d) What is the maximum number of copies that can be made using a $5 copy card?

28) Spendalot credit cards give 1% cash back on new purchases made each month. If purchases are more than $2000, then the cash back is 1.5%. Let x represent new purchases made each month.

a) Write an input-output rule for the amount of cash back on purchases that are $2000 or less.

b) Write an input-output rule for the amount of cash back on purchases that are $2000.01 or more.

The table below represents the amount of cash back received on purchases that are $2000 or less, and for purchases that are $2000.01 or more. Use this table for parts c) and d).

Input, x Purchases	**Output, y Amount of cash back**
$0 to $2000	$0.01x$
$2000.01 to credit limit	$0.015x$

c) How much cash back is received for a $1000 purchase?

d) How much cash back is received for a $3000 purchase?

1.2 SUMMARY

1. The set of **Real numbers** includes ________________ and ________________ numbers.

- __________________ numbers can be written as a quotient of two integers and include:
  - ✓ ___________________ numbers → {1, 2, 3, 4,…}
  - ✓ ___________________ numbers → {0, 1, 2, 3, 4,…}
  - ✓ ___________________ → {…–3, –2, –1, 0, 1, 2, 3,…}
- __________________ numbers cannot be written as a quotient of two integers.

2. In the expression $2x-5$:

- ________ is the variable
- ________ is the numerical coefficient of ___
- ________ is the constant.

3. Match the word in Column 1 with the operation in Column 2. Some operations have more than one match.

Column 1	**Column 2**
Product	Addition $(+)$
Less than	Subtraction $(-)$
Quotient	Multiplication $(\times)$
More than	Division $(\div)$
Decreased by	
Sum	
Percent of	

4. For the input-output table on the right, the output is one more than four times the input.

a) Write the input-output rule.

b) Complete the table.

Input, x	**Output, y**
1	5
2	9
3	
4	

5. Translate the following phrases into algebraic expressions. Use x as the input variable.

a) The difference between four and the input

b) The product of the input and three

c) The quotient of the input and eight

d) Five less than twice the input

1.3 Rectangular Coordinate System and Graphs

The Rectangular Coordinate Plane

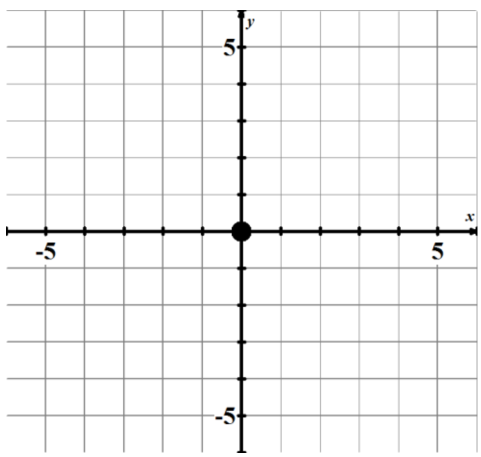
Ordered Pairs

We are now interested in studying graphs of some of the rules and tables we have created. We know that our input-output tables have two columns, one value for each variable. We call these pairs of values **ordered pairs.** Since we have a pair of values to graph, we must have a pair of axes (horizontal and vertical number lines) upon which the values can be located. These two lines form what is called a **rectangular (or Cartesian) coordinate system.** They also determine a plane.

A **plane** is a flat surface, and a result from geometry states that through any two intersecting lines (such as the axes), exactly one plane (flat surface), may be passed. If we are dealing with a linear equation in the two variables x and y, we sometimes say we are graphing the equation using a rectangular coordinate system, or that we are graphing the equation in the xy-plane.

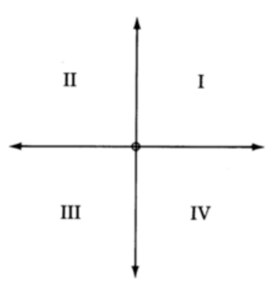
Origin

We draw the horizontal axis and vertical axis so they are perpendicular to each other and so that they intersect each other at their 0's. This point is called the **origin.**



Quadrant

Notice that the two intersecting coordinate axes divide the plane into four equal regions. Since there are four regions, we call each one a **quadrant** and number them counterclockwise using Roman numerals.



Recall that when we first studied the number line we observed the following:

For each real number there exists a unique point on the number line, and for each point on the number line we can associate a unique real number.

We have a similar situation for the plane.

For each ordered pair (x, y) there exists a unique point in the plane, and to each point in the plane we can associate a unique ordered pair (x, y) of real numbers.

Coordinates of a Point

The numbers in an ordered pair that are associated with a particular point are called the **coordinates of the point.** The **first number** in the ordered pair expresses the point's horizontal distance and direction (left or right) from the origin. The **second number** expresses the point's vertical distance and direction (up or down) from the origin.

A **positive number** means a direction to the **right or up**. A **negative number** means a direction to the **left or down**.

Plotting Points

Since points and ordered pairs are so closely related, the two terms are sometimes used interchangeably. The following two phrases have the same meaning:

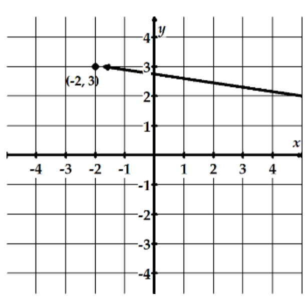
1. Plot the point (x, y).
2. Plot the ordered pair (x, y).

Both phrases mean: Locate, in the plane, the point associated with the ordered pair (x, y) and draw a point at that position.

Example 1:

Plot the point (–2, 3) on a rectangular coordinate plane.

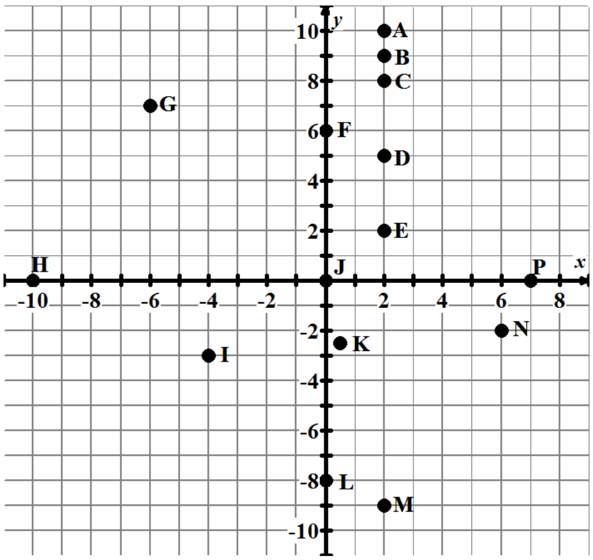
Answer:



This point is located 2 units to the left of the origin and 3 units up from the origin.

Practice Exercise: Writing the coordinates of a point

Write the coordinates of each point (in alphabetical order) plotted on the graph below.



A	I
B	J
C	K
D	L
E	M
F	N
G	P
H	

Four Representations of a Rule

So far, we have worked with the verbal (words), symbolic (written in mathematical symbols) and numerical (input-output table) representations of a rule or formula. Now we add a fourth way to represent a rule: graphically. To make the graph, we will need to add a third column to each input-output table and list the resulting (x, y) ordered pairs.

The following example illustrates the four representations of a rule.

Example 2:

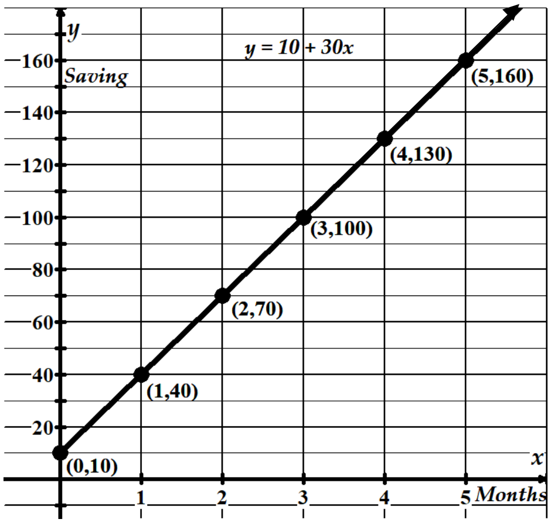
Verbal: You started this year with $10 in a savings account, and you continue to save an additional $30 per month.

Symbolic: If we let x represent the number of months and y represent the amount of money in your savings account, the equation $y = 10 + 30x$ represents this situation.

Numerical: The following input-output table is completed for the rule $y = 10 + 30x$.

# of Months, x	Amount of money in your savings account, y	Ordered Pairs (x, y)
0	10 + 30(0) = 10	(0, 10)
1	10 + 30(1) = 40	(1, 40)
2	10 + 30(2) = 70	(2, 70)
3	10 + 30(3) = 100	(3, 100)
4	10 + 30(4) = 130	(4, 130)
5	10 + 30(5) = 160	(5, 160)

Graphical: Before plotting the points in the last column and connecting them, you will need to think about the scale for the axes. The table will be helpful. The x scale ranges from 0 to 5. Notice where the tick marks are placed on the horizontal axis so that the numbers are equally spaced apart. To include the output values in the table, the y scale should range from 0 to 160.



Helpful Tip: When making a graph**, be sure to include:**

1. An appropriate scale. Use the input-output table to help you select the numbers for the horizontal axis and the numbers for the vertical axis. Try to make a scale that allows you to plot several points exactly. (In Example 2 above, the points (1, 40), (3, 100) and (5, 160) are plotted exactly.) One way to estimate finding an appropriate vertical axis scale is to take the difference of the highest output number and the lowest output number, and then divide by the number of tick marks for input values.
2. A consistent scale, which is a scale that increases with equal increments. This means that every line of the graph paper must represent the same amount. For example, the scale could be in steps of 1, or in steps of 5, or even in steps of 1000. Use the double slash (//) when there is a break in the scale. The double slash should only be used between 0 and the first number of a scale with equal increments.
3. The label for what each axis represents (x is months, y is savings), as well as the rule.
4. Connect the points (use a ruler for a linear graph) and extend the graph, if appropriate.

Interpreting Points on a Graph

Practice Exercises: Interpreting points on a graph

1) The ordered pair (3, 100) on the graph means that after _____ months, the total amount in the savings account is _________.

2) What is the meaning of the ordered pair (0, 10)?

3) Use the rule to determine how much money will be in your savings account after 1 year.

Example 3:

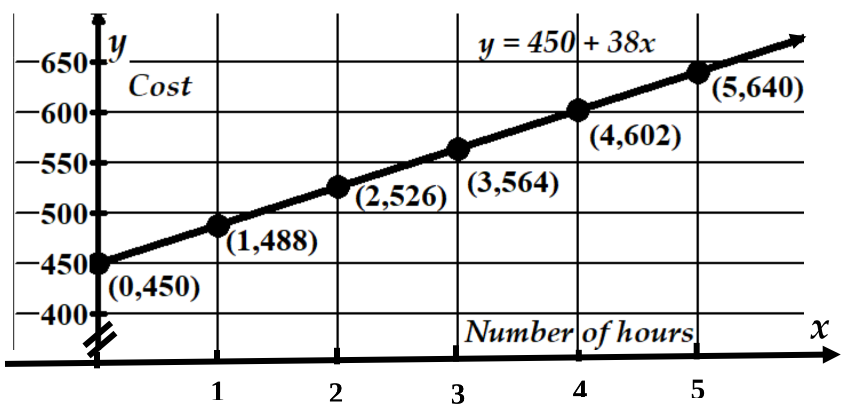
Verbal: Your cousin recently hired a contractor to do some necessary repair work. The contractor gave a quote of $450 for materials and supplies plus $38 an hour for labor.

Symbolic: If we let x represent the number of hours of labor and y represent the total cost of the contractor's labor for that number of hours, the equation $y = 450 + 38x$ represents this situation.

Numerical: Complete the following input-output table for the rule $y = 450 + 38x$.

# of Hours, x	Contractor's cost, y	Ordered Pairs (x, y)
0		
1		
2		
3		
4		
5		

Graphical: Use the table to help determine an appropriate scale for each axis. The x values are from 0 to 5, so that is easy. But the y values range from 450 to 640. In order to focus on those numbers, we should begin the scale at 450 rather than at 0. If we begin the scale at a number other than 0, we need to show there is a break in the scale by inserting a break (//) symbol. Next we decide on a scale that increases with equal increments. In the graph below the vertical scale is in steps of 50, that is, each line on the vertical axis represents an increase of 50.



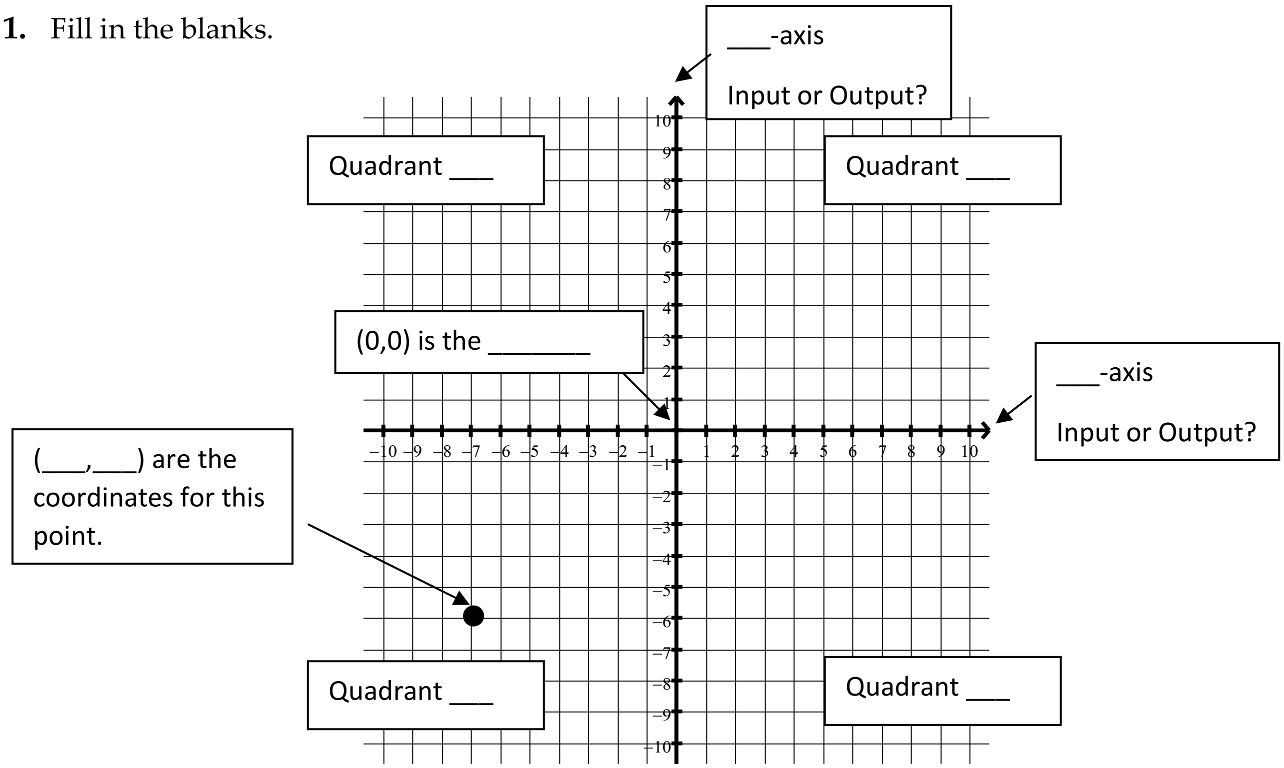
Practice Exercises: Interpreting points on a graph

4) The ordered pair (2, 526) on the graph means that after _____ hours of labor, the total cost of the contractor's job is ________.

5) What is the meaning of the ordered pair (5, 640)?

1.3 SUMMARY

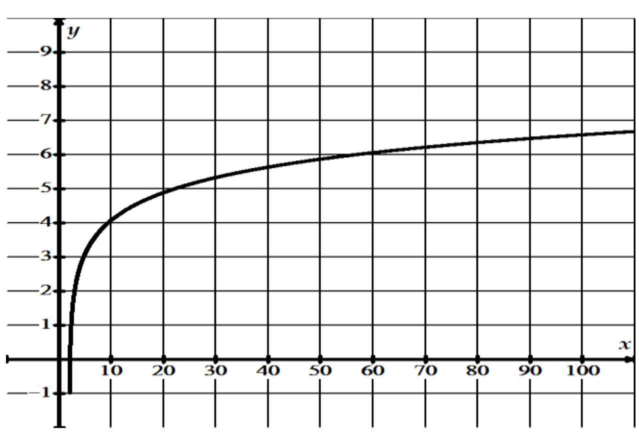
1. Fill in the blanks.



2. An ________________ (*x*, *y*) contains two real numbers that describe a __________ on a coordinate plane.

- The first number, *x*, describes the ________________ distance from the origin, along the *x*-axis.
- The second number, *y*, describes the ________________ distance from the origin, along the *y*-axis.

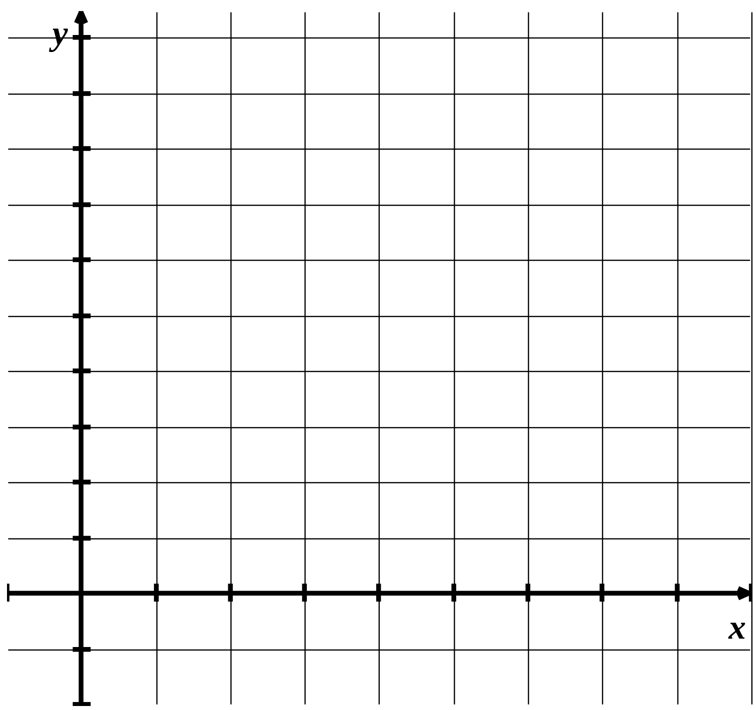
3. The __________on an axis is the amount the spaces represent between tick marks on the axis. For example, the scale of the horizontal axis for the non-linear graph below is _______.



4. The input-output table below shows the cost for driving a van x-miles. Fill-in the ordered pairs. A reasonable horizontal (x-) axis scale for the graph is _______. If you use a break, a reasonable vertical (y-) axis scale is _________.

Input, x **Miles Driven**	**Output, y** **Total Cost ($)**	**Ordered Pairs** **(x, y)**
0	100	
500	120	
1000	140	
1500	160	
2000	180	

5. Graph the ordered pairs on the axes provided, using your chosen scale. Connect the points with a straightedge. Label the axes and the tick marks.



6. The ordered pair (500, 120) on the graph means that the cost of driving _____ miles, is ________.

7. What is the meaning of the ordered pair (0, 100)?

1.4 Exponents, Fractions, and Square Root Expressions

Exponents

You may have learned that multiplication is a description for **repeated addition**. For example,

$5(3)=3+3+3+3+3.$

A natural question is "Is there a description for **repeated multiplication**?" The answer is yes. The notation that describes repeated multiplication is exponential notation.

Factors

In multiplication, the quantities being multiplied together are called factors. In repeated multiplication, all the factors are the same. In non-repeated multiplication, none of the factors are the same.

Example 1:

State whether the factors are repeated or non-repeated.

	Answers:
a) $18\cdot18\cdot18\cdot18$	Repeated multiplication of 18. All four factors, 18, are the same.
b) $x\cdot x\cdot x\cdot x\cdot x$	Repeated multiplication of x. All five factors, x, are the same.
c) $3\cdot7\cdot a$	Non-repeated multiplication. None of the factors are the same.

Exponential Notation

Exponential notation is used to show repeated multiplication of the same factor. The notation consists of using a **superscript on the factor that is repeated**. The superscript is called an exponent.

Exponential Notation: If x is any real number and n is a natural number, then

$$x^n = \underbrace{xxxx\ldots x}_{n\text{ factors of }x}$$

An exponent records the number of identical factors in a multiplication. Note that the definition for exponential notation only has meaning for natural number exponents. The notation can be extended to include other numbers as exponents, but not in this course.

Reading Exponential Notation

Definitions and Key Concepts for x^n:	
Base	x is the base.
Exponent	n is the exponent.
Power	The number represented by x^n is called a power of x.
x to the nth power	The term x^n is read as: x to the nth power, or more simply as "x to the nth."
x squared	The symbol x^2 is often read as "x squared."
x cubed	The symbol x^3 is often read as "x cubed."

Simplifying Expressions with Exponents

Example 2:

Write each of the following expressions using exponents and simplify, if appropriate.

Expanded Form:	Answers (simplified):
a) $18 \cdot 18 \cdot 18 \cdot 18$	$18^4 = 104,976$
b) $x \cdot x \cdot x \cdot x \cdot x$	x^5
c) $3 \cdot 7 \cdot a$	$21a$ (since there are no repeated factors, we cannot use exponents.)

Example 3:

Simplify each expression.

	Answers (simplified):
a) $(-2)^3$	$(-2)^3 = (-2)(-2)(-2) = -8$
b) $(-2)^2$	$(-2)^2 = (-2)(-2) = 4$
c) $x^3 \cdot x^2$	$x^3 \cdot x^2 = (x \cdot x \cdot x) \cdot (x \cdot x) = x^5$
d) $5y \cdot y^3$	$5y \cdot y^3 = 5 \cdot y \cdot (y \cdot y \cdot y) = 5y^4$ Notice that 5 is not raised to an exponent, so it is not a repeated factor.

Some expressions will have repeated factors that are both numerical coefficients and/or variables.

The following steps can be used to simplify expressions with exponents:

1) **Expand the expression so there are no exponents.**
2) **Multiply the numerical coefficients.**
3) **Write the repeated variable factors using exponents.**

Example 4:

Use the steps above to simplify each expression.

	Answers (simplified):
a) $3(-4)^2$	$3(-4)^2 = 3(-4)(-4) = 48$ Notice that the only repeated factor is – 4. It is best to use parentheses when the repeated factor is negative.
b) $3(-4x)^2$	$3(-4x)^2 = 3(-4x)(-4x) = 48x^2$ Notice that the repeated factor is – $4x$.
c) $(-2xy^3)\cdot(3x^2y^2)$	$(-2xy^3)\cdot(3x^2y^2) = -2(3)x\cdot(x\cdot x)\cdot(y\cdot y\cdot y)\cdot(y\cdot y) = -6x^3y^5$
d) $(-4x^2y)^2$	$(-4x^2y)^2 = (-4x^2y)(-4x^2y) = (-4)(-4)(x\cdot x)\cdot(x\cdot x)\cdot y\cdot y = 16x^4y^2$

Practice Exercises: Simplifying expressions with exponents

For exercises 1 – 7, simplify.

1) a) $(2y)^3$ **b)** $(-3y)^2$

2) a) $2y\cdot y\cdot y$ **b)** $3(-2)^4$

3) a) $x^3\cdot x^3$ **b)** $3(-2)^3$

4) a) $(x^3)^2$ **b)** $3(-5x)^2$

5) a) $(4x^3y)^3$ **b)** $(ab^3)(a^2b^2)$

6) a) $(-2a^3b)^2$ **b)** $(-3x^2y^3)(2xy^4)$

7) a) $(3x^4y^3)^2$ **b)** $3(-5x^4)^2$

Simplifying Fractions

In this course we work with real numbers. Although we will use a scientific calculator to obtain the result of simple calculations, you may want to complete a review of addition, subtraction, multiplication and division of integers.

When simplifying fractions, cancel the greatest common factors of coefficients and variables. A common factor can be canceled if it is in the numerator and in the denominator of the fraction.

Example 5:

Simplify the following fraction: $\dfrac{9ab}{12a}$

Answer:

Factor the numerator and the denominator and look for common factors: $\dfrac{9ab}{12a} = \dfrac{3(\cancel{3})\cancel{a}b}{4(\cancel{3})\cancel{a}} = \dfrac{3b}{4}$

Example 6:

Simplify the following fraction: $\dfrac{18x^5y^2}{4x^3y^4}$

Answer:

Writing the variables in expanded form can be helpful. Three x factors and two y factors are in the numerator and denominator, so they cancel. $\dfrac{18x^5y^2}{4x^3y^4} = \dfrac{\cancel{2}(9)\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot x\cdot x\cdot\cancel{y}\cdot\cancel{y}}{\cancel{2}(2)\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot\cancel{y}\cdot\cancel{y}\cdot y\cdot y} = \dfrac{9x^2}{2y^2}$

Helpful Tip: After canceling common factors, it is a good idea to circle the remaining factors so that you include all of them in your answer.

Practice Exercises: Simplifying Fractions

For exercises 8 – 15, simplify each of the following fractions.

8) $\dfrac{y^8}{y^4}$

9) $\dfrac{9x}{24x}$

10) $\dfrac{15ab}{5bc}$

11) $-\dfrac{12x^3y^2}{10x^3y^3}$

12) $\dfrac{\left(6x^2\right)^3}{\left(2x\right)^2}$

Hint: For exercises 13 – 15, simplify the fraction inside of parentheses first.

13) $\left(\dfrac{6x^2}{2x}\right)^2$

14) $-\left(\dfrac{3x}{x^2}\right)^2$

15) $\left(-\dfrac{x^2y}{xy^3}\right)^2$

Square Root Expressions

When we studied exponents, we noted that $4^2 = 16$ and $\left(-4\right)^2 = 16$. We can see that 16 is the square of both 4 and – 4. Since 16 comes from squaring 4 or – 4, 4 and – 4 are called the **square roots** of 16.

Principal Square Root of x: $\sqrt{x}$

If x is a positive real number, then $\sqrt{x}$ represents the positive (or principal) square root of x.

The symbol $\sqrt{\ }$ is called a radical sign.

Perfect Squares:

The following numbers are called perfect squares. When you take the square root of each number, the result is a positive number.

Whole Numbers that are Perfect Squares: {0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, …}

Note the use of the **ellipsis (the three dots)**, since there are infinitely many perfect squares. We cannot list all of the perfect squares. In fact 0.25 is a perfect square, because $(0.5)^2 = 0.25$ and therefore $\sqrt{0.25} = 0.5$.

Meaningful Square Root Expressions

- Remember that if we take the square root of a number that is not a perfect square, we get an **irrational number** (a non-repeating, non-terminating decimal). For example, use your calculator to find $\sqrt{50}$. We say that $\sqrt{50} \approx 7.07$, rounded to the nearest hundredth.
- Since we know that the square of any real number is a positive number or zero, we can see that expressions such as $\sqrt{-16}$ do not describe real numbers. There is no real number that can be squared and produce −16. For $\sqrt{x}$ to be a real number, then it is essential that $x \geq 0$. In our study of algebra, we will assume that all variables and all expressions in **radicands** (expressions under the radical sign) represent nonnegative numbers (numbers greater than or equal to zero).

Simplifying Square Root Expressions

To multiply two square root expressions, we use the product property for square roots.

The Product Property

$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$$

To simplify the square root of a quotient, we use the quotient property for square roots.

The Quotient Property

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Example 7:

Use properties of square roots to simplify each of the following expressions. Do not use a calculator. Answers must be exact.

		Answers
a)	$\sqrt{2}\sqrt{8}$	$\sqrt{2}\sqrt{8}=\sqrt{2\cdot 8}=\sqrt{16}=4$
b)	$\left(2\sqrt{3}\right)^2$	$\left(2\sqrt{3}\right)^2=\left(2\sqrt{3}\right)\left(2\sqrt{3}\right)=2\cdot 2\cdot\sqrt{3}\cdot\sqrt{3}=4\cdot\sqrt{9}=4\cdot 3=12$
c)	$\sqrt{\dfrac{9}{16}}$	$\sqrt{\dfrac{9}{16}}=\dfrac{\sqrt{9}}{\sqrt{16}}=\dfrac{3}{4}$
d)	$\dfrac{\sqrt{50}}{\sqrt{2}}$	$\dfrac{\sqrt{50}}{\sqrt{2}}=\sqrt{\dfrac{50}{2}}=\sqrt{25}=5$

Practice Exercises: Simplifying Square Root Expressions

For exercises 16 – 23, use properties of square roots to simplify. Do not use a calculator. Answers must be exact.

16) $\left(\sqrt{5}\right)^2$

17) $-\left(\sqrt{5}\right)^2$

18) $\left(-\sqrt{5}\right)^2$

19) $\left(\sqrt{25}\right)^2$

20) $\sqrt{1.2}\sqrt{0.3}$

21) $\sqrt{\dfrac{36}{81}}$

22) $\sqrt{2}\sqrt{18}$

23) $\dfrac{\sqrt{108}}{\sqrt{3}}$

1.4 SUMMARY

1. Fill in the blanks:

m is called the ____________

$$a^m$$

a is called the ________

2. To multiply or divide expressions with exponents – focus on like bases.

Simplify each of the following:

a) $\dfrac{9x^3y}{3x^2y^3} =$ **b)** $\left(-5a^4b^2\right)\left(2a^2b\right) =$ **c)** $\left(a^3\right)^2 =$

3. **Exponent Rules**

Expanding exponential expressions with parentheses:

$(ab)^3 =$	and $\left(\dfrac{a}{b}\right)^4 =$

Rules for square roots:

$\sqrt{ab} =$	and $\sqrt{\dfrac{a}{b}} =$

4. **Multiple Choice**: Circle one equivalent expression in each of the following.

I. $(-6xy)^2 =$

a) $36xy$ **b)** $-36x^2y^2$ **c)** $36x^2y^2$ **d)** $6x^2y^2$

II. $\dfrac{\sqrt{48}}{\sqrt{3}} =$

a) 16 **b)** 4 **c)** 6 **d)** none of these

1.5 Order of Operations, Properties of Real Numbers, and Simplifying Expressions

Order of Operations

Evaluate: $10-2(4-7)+24\div(2)(3)$.

If several of your friends who do not know algebra try this problem, they may get different answers. However, there is only one correct answer, and it is 52. If you follow the order of operations below you should get 52, and if you enter the problem into your calculator exactly as it is written, the result is 52. Calculators and computers are programmed to follow the order of operations.

Order of Operations:

1) **Perform all operations inside grouping symbols beginning with the innermost set.**
2) **Perform all exponent and square root operations as you come to them, moving from left-to-right.**
3) **Perform all multiplications and divisions from left-to-right.**
4) **Perform all additions and subtractions from left-to-right.**

What are grouping symbols?	**Examples:**
Parentheses	$2(3+4)=2(7)=14$
Brackets	$-[5-8]=-[-3]=3$
Braces	$\{5-8\}-3=\{-3\}-3=-6$
Absolute Value	$\lvert 5-8\rvert=\lvert -3\rvert=3$
Fractions	$\frac{2+6}{2+8}=\frac{8}{10}=\frac{4}{5}$
Expressions under a square root	$\sqrt{9+16}=\sqrt{25}=5$

Therefore to evaluate $10-2(4-7)+24\div(2)(3)$ **using the order of operations:**

$10-2(4-7)+24\div(2)(3)$

$10-2(-3)+24\div(2)(3)$ simplify what is inside parenthesis first: $4-7$ is -3

$10+6+12(3)$ multiply or divide from left to right

$10+6+36=52$ add or subtract from left to right

Practice Exercises: Order of Operations

For exercises 1 – 10, use the order of operations to evaluate each expression:

1) $\dfrac{-10(-5)}{-3+1}$

2) $2(3-5)^2$

3) $2\left|3^2-15\right|$

4) $10-5(3-1)$

5) $(10-5)(3-1)$

6) $\dfrac{-10-5+1}{-2}$

7) $\sqrt{5^2-3^2}$

8) $2(5^2-3)$

9) $11-5\cdot3-1$

10) $11-(5+3)^2-1$

Properties of Real Numbers

Some mathematical problems can be calculated in your head, without a calculator. The answer to: $23+14+7+6$ can be obtained by changing the order and grouping of the numbers. One way to look at the problem is: $23+14+7+6=(23+7)+(14+6)=30+20=50$ The properties of real numbers that permit changing the order and grouping of numbers follow.

The Commutative Properties of Addition and Multiplication

The **commutative** properties tell us that two numbers can be added or multiplied **in any order** without affecting the result.

Let a and b represent real numbers.

Commutative Property of Addition:	$a+b=b+a$
Commutative Property of Multiplication:	$a(b)=b(a)$

Example 1:

The following are examples of the commutative properties.

a) $2+4=4+2$ Both equal 6.
b) $3+x=x+3$ Both represent the same sum.
c) $4(5)=5(4)$ Both equal 20.
d) $x(7)=7(x)$ Both represent the same product. Note that $7(x)$ is usually written as $7x$.
e) $4(x+1)=(x+1)4$ Both represent the same product.
f) $(x+1)(x-7)=(x-7)(x+1)$ Both represent the same product.

The Associative Properties of Addition and Multiplication

The **associative** properties tell us that when adding or multiplying three or more numbers, they can be **grouped** in any way, without affecting the result.

Let a and b represent real numbers.

Associative Property of Addition:	$a+(b+c)=(a+b)+c$
Associative Property of Multiplication:	$a(bc)=(ab)c$

Example 2:

The following are examples of the associative properties. Notice that the order of the numbers has not changed, but the grouping has changed.

a)
$$\begin{aligned} 2+(3+4) &= (2+3)+4 \\ 2+(7) &= (5)+4 \\ 9 &= 9 \end{aligned}$$

b)
$$\begin{aligned} 2(3\cdot 4) &= (2\cdot 3)4 \\ 2(12) &= (6)4 \\ 24 &= 24 \end{aligned}$$

Practice Exercises: Commutative and Associative Properties

For exercises 11 – 15, fill in the $(\ )$ to make each statement true. State which property you used.

11) $7+(2+4)=(\quad)+4$

12) $4(2)=2(\ )$

13) $2(3x)=(\ )x$

14) $x+4=4+(\ )$

15) $(x-5)(x+4)=(x+4)(\quad)$

The Distributive Property

Suppose you are on line at the bookstore, and you plan to purchase four notebooks. Each notebook is \$5.25 and you want to pay cash. Without using a calculator, how can you calculate the cost of the notebooks before tax is added?

The problem is to multiply $4(5.25)$. One way to do this is by thinking of (5.25) as (5 + 0.25). Then we have: $4(5.25)=4(5+0.25)=4(5)+4(0.25)=20+1=\$21$. This process illustrates the distributive property, which involves both multiplication and addition.

Let *a* and *b* represent real numbers.

Distributive Property of Multiplication over Addition: $a(b+c)=ab+ac$

Example 3:

The following examples show how to use the distributive property to rewrite each expression.

a)
$$\begin{aligned} 2(3+4) &= 2(3)+2(4) \\ &= 6+8 \\ &= 14 \end{aligned}$$

b)
$$\begin{aligned} 2(x+4) &= 2(x)+2(4) \\ &= 2x+8 \end{aligned}$$

Practice Exercises: Distributive Property

For Exercises 16 – 18, use the distributive property to rewrite each expression. Simplify where appropriate.

16) $2(x+3y-4z)$

17) $5-2(x-7)$

18) $5-2(4-7)$

Terms

The distributive property is frequently used when simplifying algebraic expressions. In Example 3 part b) above, $2(x+4)$ is the product of two factors: 2 and $(x+4)$.

After applying the distributive property, we now have an equivalent expression which is a sum. In an algebraic expression, the quantities joined by "+" or "– " signs are called **terms.**

The Difference between Terms and Factors

An important concept that all algebra students must be aware of is the difference between terms and factors.

Terms are parts of sums and are therefore joined by addition (or subtraction) signs.

Factors are parts of products and are therefore joined by multiplication (or division) signs.

Practice Exercises: The Difference between Terms and Factors

For exercises 19 – 20, identify terms or factors as appropriate.

19) List the terms, separated by a comma, in the expression: $4x^2-2xy-y+1$.

20) List the factors, separated by a comma, in the expression: $4x(x-3)$.

Like Terms

Terms whose variable parts, including the exponents, are identical are called **like terms**. Like terms is an appropriate name since terms with identical variable parts and different numerical coefficients represent different amounts of the same quantity. As long as we are dealing with quantities of the same type we can combine them using addition and subtraction.

Simplifying an Algebraic Expression

An algebraic expression can be simplified by combining like terms.

Example 4:

Combine like terms.

Examples with Answers:	**Solution:**
a) 6 pens + 4 pens = 10 pens.	6 and 4 of the same type give 10 of that type.
b) 6 pens + 4 pens + 2 pencils = 10 pens + 2 pencils.	6 and 4 of the same type give 10 of that type. Thus we have 10 of one type and 2 of another type.
c) Suppose we let "x" represent pen. Then $6x + 4x = 10x$.	6 and 4 of the same type give 10 of that type.
d) Suppose we let "x" represent pen and "y" represent pencil. Then $6x + 4x + 2y = 10x + 2y$.	6 and 4 of the same type give 10 of that type. Thus we have 10 of one type and 2 of another type.

Simplifying expressions with parentheses

To simplify expressions with parentheses, first use the distributive property to remove parentheses. Then, combine like terms.

Example 5:

Simplify the following expression: $6x+5(x-3)$.

Answer: $6x+5(x-3)=6x+5x-15$ Use the distributive property to remove parentheses.

$11x-15$ Combine the like terms: $6x$ and $5x$.

Practice Exercises: Combining Like Terms

For exercises 21 – 30, combine like terms. Use the distributive property first, where appropriate.

21) $3x+6x+11x$

22) $5a+2b+4a-b-7b$

23) $10x^3-4x^3+3x^2-12x^3+5x^2+2x+x^3+8x$

24) $4x^2y+3xy^2-6x^2y-3xy^2$

25) $4x+9(x-3)$

26) $6(x-1)-2(x-3)$

27) $3(x-4)-(x-3)$

28) $12-6(x+3)-2x-3$

29) $4(x+3)+3(2+x+3x^2)-2x^2$

30) $8-(y+1)+y^2-2y$

Simplifying Algebraic Fractions

Remember that when simplifying algebraic fractions, we cancel common factors in the numerator and in the denominator. This can happen when the numerator and denominator only have factors, as in: $\dfrac{4x\cdot 8}{2}=\dfrac{32x}{2}=16x.$

However when the numerator has more than one term and the denominator has exactly one term, then we can **apply the distributive property**, as in: $\dfrac{4x-8}{2}=\dfrac{4x}{2}-\dfrac{8}{2}.$

We think of simplifying a fraction with one term in the denominator as applying the distributive property, because we are "distributing" the denominator to each term in the numerator.

When we rewrite the expression as two fractions, we simplify each fraction by canceling common factors in the numerator and denominator, the result is: $2x-4$.

Another way to look at this problem is realizing that dividing by 2 is the same as multiplying by $\dfrac{1}{2}$. And so: $\dfrac{4x-8}{2}=\dfrac{1}{2}(4x-8)=\dfrac{1}{2}(4x)-\dfrac{1}{2}(8)=2x-4.$

Practice Exercises: Simplifying Algebraic Fractions

For exercises 31 – 35, simplify each expression.

31) $\dfrac{-12xy}{4x}$ **32)** $\dfrac{-12x+8}{4}$ **33)** $\dfrac{5x+15}{3}$ **34)** $\dfrac{15xy}{3y}$ **35)** $\dfrac{6x+16}{3}$

1.5 SUMMARY

1. **Explain the meaning of the letters representing the order of operations, as listed below.**
   P:______________________ or other grouping symbols, i.e.________________________________
   E:______________________
   $\left.\begin{matrix}\text{M}\\\text{D}\end{matrix}\right\}$:___________________ or __________________ from ________ to ___________
   $\left.\begin{matrix}\text{A}\\\text{S}\end{matrix}\right\}$:___________________ or __________________ from ________ to ___________

2. **Use the order of operations to evaluate each of the following:**

   a) $2\left|3-5^2\right|$ **b)** $10-(3-1)$

3. **Factors vs. Terms** The difference between factors and terms is that factors are ___________ (added or multiplied) and terms are _____________ (added or multiplied).

   Example:
   The expression $2x-3xy$ has __________ terms. The first term is______________.
   The factors of the first term are ____________. The second term is __________________.
   The factors of the second term are: _________________________.

4. To **combine like terms**, __________ (add, multiply, keep) the coefficients and _________(add, multiply, keep) the variable along with its exponent.

   Example: **a)** $-5x+10y+2x+3y=$ **b)** $x^2+2x+7x^2-5x=$

5. **Properties of Real Numbers**

   Associative Property (Regroup): $a+(b+c)=(a+\_\_)+\_\_$ **OR** $a(b\cdot c)=(\_\_\cdot b)\cdot\_\_$
   Commutative Property (Reorder): $a+b=$ _______ **and** $ab=$ _______
   Distributive Property (Multiplication Across Addition): $a(b+c)=$ _____+ _____

6. **Simplification of Fractions:**
   - Fraction with a **single term** in numerator and denominator – ______________ common factors. $\dfrac{-18ab}{12bc}=$
   - Fractions with **more than one term in the numerator and one term in the denominator** – use the _______________ property to divide each term by the denominator; then cancel common factors in each term. $\dfrac{16x-10}{4}=$

1.6 Polynomials and Evaluating Expressions

Simplifying Polynomial Expressions

We will continue to simplify expressions by introducing some vocabulary associated with polynomials.

Definitions and Key Concepts	
Polynomial	An algebraic expression composed of the sum of terms containing variable(s) raised to a non-negative integer exponent.
Monomial	A polynomial consisting of one term.
Binomial	A polynomial consisting of two terms.
Trinomial	A polynomial consisting of three terms.
Leading term	The term that contains the highest power of the variable in a polynomial.
Leading coefficient	The coefficient of the leading term.
Constant term	A number with no variable factors. A term whose value never changes.
Degree	The highest exponent in a polynomial.
Descending Order	Polynomials are generally written in descending order, that is, terms are written so that exponents are decreasing.

Practice Exercise: Polynomial vocabulary

For exercise 1, complete the table. (Write monomial, binomial or trinomial in the **Name** column.)

1)

Polynomial	Name	Leading coefficient	Constant term	Degree
$2x^5 - x^2 + 3$				
$5 - x^2$				
$2x - 1$				
$-8x^3$				

Adding, Subtracting and Multiplying Polynomials

Addition and Subtraction of Polynomials

To add and subtract polynomials, simply combine like terms. You may need to use the distributive property first, so that the expression consists of all terms.

Multiplication of Polynomials:

We have already seen how to multiply by monomials, as in Example 1.

Example 1:

Multiply each of the following and then simplify.

Example:	**This is the product of:**	**Answers:**
$2x^4\left(-3x^3\right)$	Two monomials	Multiply numerical coefficients and write repeated factors using exponents. $2x^4\left(-3x^3\right)=2(-3)(xxxx)(xxx)=-6x^7$
$2x\left(5-3x^2\right)$	A monomial and a binomial	Use the distributive property: $2x\left(5-3x^2\right)=2x(5)-2x\left(3x^2\right)=10x-6x^3=-6x^3+10x$
$2x\left(5-x-3x^2\right)$	A monomial and a trinomial	Use the distributive property: $2x\left(5-x-3x^2\right)=2x(5)-2x(x)-2x\left(3x^2\right)=-6x^3-2x^2+10x$

Practice Exercises: Simplifying Polynomial Expressions

For exercises 2 – 10, simplify each expression.

2) $\left(4x^2+2x-1\right)+3\left(x^2-2x\right)$

3) $6(x+3)-2(x-3)$

4) $3x-x(x-1)$

5) $\left(x^3-2x-1\right)-\left(5x^3-x^2-2x+1\right)$

6) $2x\left(x^2-3x-1\right)$

7) $2x\left(x^3-3x-4\right)$

8) $3x^3\left(-2x^2\right)$ *Hint:* in this example, you need to find the product of two monomials. You do **not** use the distributive property to simplify.

9) $-\left(x-1\right)$ *Hint:* in this example, you need to distribute the negative that is in front of the parentheses. The negative sign to be distributed is often thought of as the factor: – 1.

10) $2\left(2x^2-3x+2\right)-3\left(x^2-2x+1\right)$

Next we look at how to use the distributive property to multiply two binomials. You may have heard this referred to as FOIL.

To multiply two binomials, multiply every term of one binomial by every term of the other binomial. This is often referred to as FOIL, which means add the following products in this specific order: the First terms, the Outer terms, the Inner terms, and the Last terms.

Example 2:

Multiply: $(a+6)(a+3)$.

Answer:

$$\begin{aligned}(a+6)(a+3) &= a\cdot a + a\cdot 3 + 6\cdot a + 6\cdot 3\\ &= a^2+3a+6a+18\\ &= a^2+9a+18\end{aligned}$$

F: $\boldsymbol{a\cdot a}$
O: $\boldsymbol{3\cdot a}$
I: $\boldsymbol{6\cdot a}$
L: $\boldsymbol{6\cdot 3}$

With some practice, the second and third terms can be combined mentally.

Practice Exercises: Multiplying Two Binomials

For exercises 11 – 20, multiply the binomials in each expression.

11) $(x+5)(x+3)$

12) $(x-5)(x-3)$

13) $(x+5)(x-3)$

14) $(x-5)(x+3)$

15) $(x-1)(x+1)$

16) $(x+3)^2$ (*Hint*: expand first)

17) $(2x-3)(x-4)$

18) $(2x-1)(3x+4)$

19) $(2x-3)^2$

20) $(2x+3)(2x-3)$

Evaluating Expressions

Most of the expressions that we evaluate will contain variables. We know that a variable represents an unknown quantity. Therefore, any expression that contains a variable represents an unknown quantity. For example, if the value of x is unknown, then the value of $2x+3$ is unknown. The value of $2x+3$ depends on the value of x.

Numerical evaluation is the process of determining the numerical value of an algebraic expression by replacing the variables in the expression with specified numbers.

Helpful Tip: It is a good idea to use parentheses when replacing the variable with a specified number.

Example 3:

Evaluate:	Solution:	Answer:
$2x+3$ for $x=2$	$2(2)+3=4+3=7$	7
$2x+3$ for $x=-2$	$2(-2)+3=-4+3=-1$	–1
x^2 for $x=5$	$(5)^2=(5)(5)=25$	25
x^2 for $x=-5$	$(-5)^2=(-5)(-5)=25$	25
$-x^2$ for $x=-5$	$-(-5)^2=-(-5)(-5)=-25$	– 25
$\frac{64}{x^2}$ for $x=-4$	$\frac{64}{(-4)^2}=\frac{64}{16}=4$	4
$x^2-12x+1$ for $x=-5$	$(-5)^2-12(-5)+1=25+60+1=86$	86
a^2-b^2 for $a=-2$ and $b=-3$	$(-2)^2-(-3)^2=4-9=-5$	– 5
$\sqrt{\frac{32}{b^3}}$ for $b=2$	$\sqrt{\frac{32}{(2)^3}}=\sqrt{\frac{32}{8}}=\sqrt{4}=2$	2

Helpful Tip: Note that when squaring a number, the result is always positive. According to order of operations, evaluating exponents is done before multiplying by the negative. Therefore when evaluating $-x^2$ for $x=-5$ the –5 is squared first. The negative sign in front of x^2 indicates that the numerical coefficient is actually –1, so the final answer is multiplied by –1 and is therefore negative.

Remember: For any real number *x:*

- x^2 is always positive
- $-x^2$ is always negative

Practice Exercises: Evaluating Expressions

For exercises 21 – 26, evaluate each expression by substituting the specified numbers

21) $6-x$ for $x=-8$.

22) $2\sqrt{x-1}$ for $x=82$.

23) $\sqrt{4x^2}$ for $x=3$.

24) $a(b-c)$ for $a=-5$, $b=1$ and $c=-4$.

25) $-x^2-12x+1$ for $x=-5$.

26) $5-3|4-x|$ for $x=-8$.

For exercises 27 – 28, complete the tables below and **plot the ordered pairs on graph paper**. Connect the points and decide whether each graph should be extended.

27)

x	$y=x^2-2x-1$	(x,y)
-2		
-1		
0		
1		
2		
3		
4		

28)

x	$y=x^3-4x$	(x,y)
-3		
-2		
-1		
0		
1		
2		
3		

1.6 SUMMARY

1. A **polynomial** is an expression containing one or more terms being ____________ or subtracted. The exponents on the variables in each term must be _______________ integers.

- **Monomial** - a polynomial expression containing _____________ term.
- **Binomial** - a polynomial expression containing _____________ terms.
- **Trinomial** - a polynomial expression containing _____________ terms.

<u>Example</u>: The polynomial $-5x^3+x-1$ has ______ terms. The leading term is _____ and the coefficient of the leading term is ____. The constant term is ___.

2. **To add or subtract polynomials**, combine ______________ terms.

<u>Example</u>:
$$\left(4x^2-3x+2\right)-\left(x^2-2x+4\right)$$
$$=4x^2-3x+2-x^2 \_\_\_\; 2x \_\_\_\; 4=$$

3. **To multiply a monomial by a polynomial** use the ______________________property.

<u>Example</u>: $-3x^4\left(x^3-20x+10\right)=$

4. **To multiply binomials** use distribution twice or FOIL.

<u>Example</u>:

$$(x+4)(x-5)=x^2-5x+4x-20= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

F O I L

5. To **evaluate an algebraic expression** for a given value means to ____________________ the value of the variable each time the variable occurs.

Example: Evaluate $-x^2+3x-2$, for $x=-2$

Solution: $-(\_\_\_\_\_)^2+3(\_\_\_\_\_)-2=-\_\_\_\_\_+\_\_\_\_\_-2=$

Replace each x with –2

Remember to put the (–2) in parentheses!!

Follow order of operations to simplify.

1.7 Functions

Relations and Functions

The weight of a growing child increases with time. The amount of sales tax changes as the price of an item increases. In each case, one quantity depends on another. There is a relationship between the two quantities that we can describe, analyze, and use to make predictions. In this section we will investigate such relationships, and decide if they are relations or functions.

Relations

A **relation** is a set of ordered pairs. The set consisting of the first components of each ordered pair is called the **domain** and the set consisting of the second components of each ordered pair is called the **range**. Consider the following set of ordered pairs. The first numbers in each pair are the first five natural numbers. The second number in each pair is twice the first number.

$$\{(1, 2), (2, 4), (3, 6), (4, 8), (5, 10)\}$$

The domain is $\{1, 2, 3, 4, 5\}$.The range is $\{2, 4, 6, 8, 10\}$.

Note that each value in the domain is also known as an input value, or **independent variable**, and is often labeled with the lowercase letter x. Each value in the range is also known as an output value, or **dependent variable**, and is often labeled lowercase letter y.

Definition of a Function

A function f is a relation that assigns a single element in the range to each element in the domain. In other words, when looking at the input-output table, **no x-values are repeated**.

For our example that relates the first five natural numbers to numbers double their values, this relation is a function because each element in the domain, $\{1, 2, 3, 4, 5\}$, is paired with exactly one element in the range, $\{2, 4, 6, 8, 10\}$.

Now let's consider the following set of ordered pairs that relates the words "even" and "odd" to the first five natural numbers: $\{(\text{odd}, 1), (\text{even}, 2), (\text{odd}, 3), (\text{even}, 4), (\text{odd}, 5)\}$

Notice that each element in the domain, {even, odd} is not paired with exactly one element in the range, $\{1, 2, 3, 4, 5\}$. For example, the term "odd" corresponds to three values from the range, $\{1, 3, 5\}$ and the term "even" corresponds to two values from the range, $\{2, 4\}$. This violates the definition of a function, so this relation is not a function.

Example 1:

Figure 2 below compares relations that are functions and not functions. State the domain and range for each relation or function.

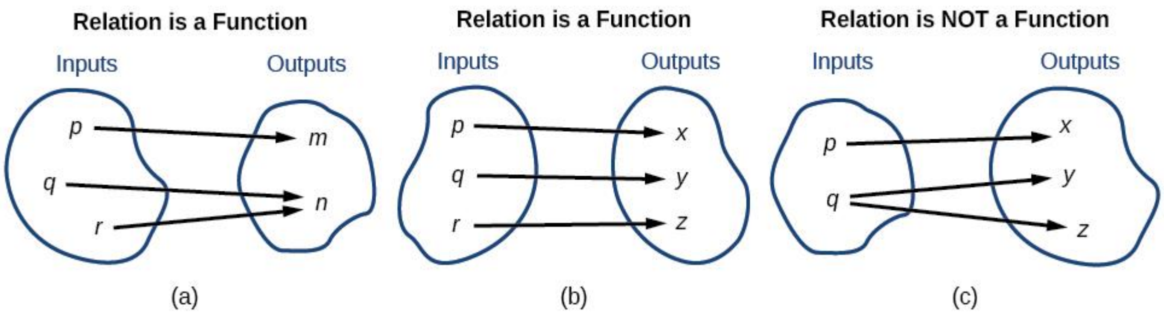


Figure 2.

<u>Answers</u>:

(a) The domain is $\{p, q, r\}$ and the range is $\{m, n\}$. This relationship is a function because each input is associated with a single output. Note that input q and r both give output n.

(b) The domain is $\{p, q, r\}$ and the range is $\{x, y, z\}$. This relationship is also a function. In this case, each input is associated with a single output.

(c) The domain is $\{p, q\}$ and the range is $\{x, y, z\}$. This relationship is not a function because input q is associated with two different outputs.

Numerical Representation of a Function

Another way to determine whether a relation is a function, is by putting the ordered pairs into an input-output table. In a function, each **input element should only be listed once**, and have exactly one corresponding output.

Example 1 Solution (continued) Using Tables:

a)

Inputs	Outputs
p	m
q	n
r	n

b)

Inputs	Outputs
p	x
q	y
r	z

c)

Inputs	Outputs
p	x
q	y
q	z

By looking at the table, it is clear that part c) is not a function because the input "q" is <u>repeated in the input column</u> and has two different corresponding outputs.

Practice Exercises: Identifying Functions

For exercises 1 – 4, decide whether each verbal explanation below is a relation or a function. Explain your answer. *Hint:* Write some ordered pairs. If an input value is repeated with two or more different corresponding output values, then it is a relation and not a function.

1) The height (in feet), y, of a golf ball x seconds after being hit is given by the equation $y = -16x^2 + 80x$.

2) The input is each student in this class, the output is the birthday of that student.

3) The input is September 1, the output is any Brookdale student born on that day.

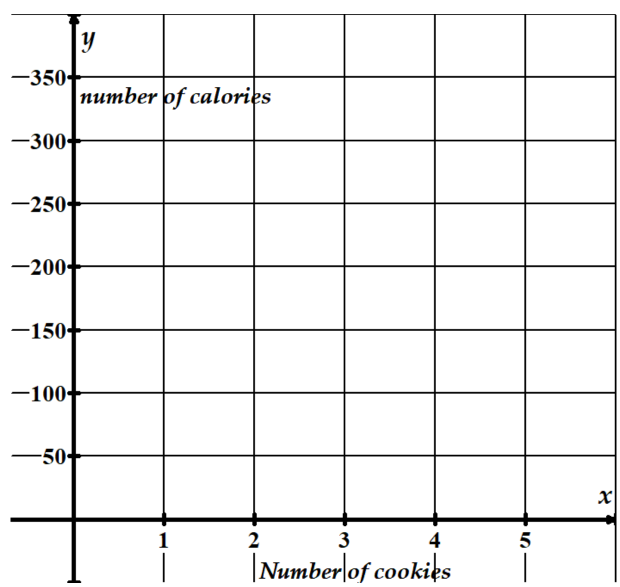
4) There are about 80 calories in one chocolate chip cookie. If x cookies are eaten, write a rule that determines how many calories are consumed, y. Is this rule a function?

Graphical Representation of a Function

Complete the table below for Practice Exercise 4. Then plot and connect the points on the given graph. Make sure to label the axes.

Number of cookies, x	Number of calories, y	(x, y)
1		
2		
3		
4		

The graph shows that for this function, each serving of cookies yields a specific number of calories. For each input there is exactly one output.



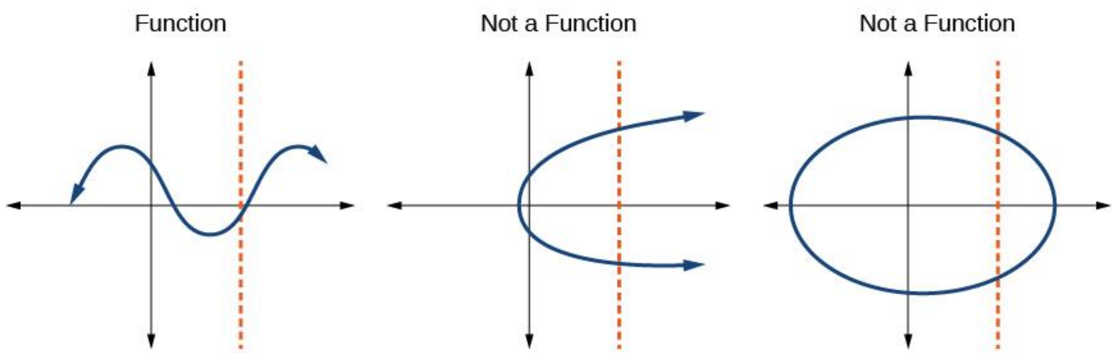
The Vertical Line Test

The **vertical line test** can be used to determine whether a graph represents a function. If we can draw any vertical line that intersects a graph more than once, then the graph does *not* define a function because a function has only one output value for each input value.

How to Use the Vertical Line Test to determine if the graph represents a function.
1) Inspect the graph to see if any vertical line drawn would intersect the curve more than once.
2) If there is any such line, determine that the graph does not represent a function.

Example 2:

Use the Vertical Line Test to determine if each graph is a function.



Answers:

The first graph is a function, because any vertical line crosses the graph in exactly one point.

The second and third graphs are not a function because there is a vertical line that crosses each graph in more than one point.

Function Notation

Once we determine that a relationship is a function, we need to display and define the functional relationships using a standard notation, so that we can understand and use them.

The notation $y = f(x)$ defines a function named f. This is read as "y is a function of x."
The letter x represents the input value, or independent variable. The letter y, or $f(x)$ represents the output value, or dependent variable.
The letters f, g, and h are often used to represent functions.

How to Write a Rule using Function Notation

We saw that in Practice Exercise 4, the function rule is $y = 80x$. In function notation, this rule is written as $f(x) = 80x$ When using function notation to represent a rule, simply replace "$y =$" with "$f(x) =$".

Example 3:

Each of the following rules is a function. The second column gives information about the type of function. In column 3, write each rule using function notation.

Rules:	Type of Function	Answer: Function notation
$y = x^2 - 2x - 1$	Quadratic Function – polynomial, degree 2.	$f(x) = x^2 - 2x - 1$
$y = x^3 - 4x$	Cubic Function – polynomial, degree 3.	$f(x) = x^3 - 4x$
$y = \frac{32}{x^2}$	Rational Function – is a fraction with the variable in the denominator.	$f(x) = \frac{32}{x^2}$
$y = 2\sqrt{x-1}$	Radical Function – has the variable under a square root.	$f(x) = 2\sqrt{x-1}$

Practice Exercises: Function Notation

For exercises 5 – 7, answer each of the following:

- **i.** Decide on the two variables, determine which variable is the input and which variable is the output.
- **ii.** Choose variable names.
- **iii.** Decide whether the output is a function of the input and, if so, write the relationship using function notation.

5) The cost to mail a package is a function of the number of ounces it weighs. The cost is \$0.20 per ounce.

6) The cost of buying ham at the deli is a function of the weight. Ham costs \$5.99/lb.

7) The cost of making copies is a function of the number of copies you are making. Each copy costs \$0.10.

Using Function Notation

Examples 4 and 5 show two different versions of the same problem: evaluating an expression. Notice how function notation in Example 5 provides a more efficient and concise way to give directions for evaluating an expression.

Example 4:

Evaluate the following expression by substituting the specified number:
$x^2 - 12x + 1$ for $x = -5$

Answer: $x^2 - 12x + 1 = (-5)^2 - 12(-5) + 1 = 86$

Example 5:

If $f(x) = x^2 - 12x + 1$, find $f(-5)$.

Answer: $f(-5) = x^2 - 12x + 1 = (-5)^2 - 12(-5) + 1 = 86$

Helpful Tip: Remember that when using function notation, x still represents the input (the domain) and $f(x)$ is used to represent the output values (the range).

Practice Exercises: Using Function Notation

For exercises 8 – 12, evaluate the given value for each function.

8) If $f(x) = x - 4$, find $f(4)$.

9) If $f(x) = 1 - 4x$, find $f(0)$.

10) If $f(x) = -x^2 - 12x + 1$, find $f(2)$.

11) If $f(x) = \dfrac{32}{x^2}$, find $f(-2)$.

12) If $f(x) = 2\sqrt{x - 1}$, find $f(10)$.

Interpreting Function Notation

Function notation can be used in applied situations, such as Example 6.

Example 6:

The cost, C, of making copies is a function of the number of copies, x, you are making. Each copy costs \$0.10. Build a table using integers from 0 to 100 in steps of 20 for the number of copies, x. Use the table to answer the following:

a) Find the cost of making 20 copies.

b) If you spend \$8.00, how many copies were made?

Answers:

Since the cost of making copies is a function of the number of copies made, the rule for this situation is: $C(x) = 0.10x$.

Notice that the third column for the ordered pairs is written in function notation.

Number of copies, x	Cost of making copies, $C(x) = 0.10x$	$(x, C(x))$
0	0	(0, 0)
20	2.00	(20, 2)
40	4.00	(40, 4)
60	6.00	(60, 6)
80	8.00	(80, 8)
100	10.00	(100, 10)

From the table, we can answer the following questions:

a) The number of copies is the input for this example. We are being asked to find the cost of making 20 copies, so $x = 20$. When we look in the x column for 20, we see that the output is 2. Note that we can find the cost of making 20 copies by using function notation: $C(20) = 0.10(20) = 2.00$. Therefore the cost of making 20 copies is \$2.

b) If the cost is \$8.00, we are given the output value, 8.00. So we look in the output (y) column, which in function notation is the $C(x)$ column. When the output is 8.00, the input is 80, therefore $x = 80$. This means that for \$8.00, 80 copies can be made.

Practice Exercises: Interpreting Function Notation

For exercises 13 – 14, consider the following situation: tuition for a semester is a function of the number of credits a student takes. Let c = the number of credits and t = the tuition for a semester. Interpret each of the following.

13) $t(12) = 5400$

14) $t(15) = 6750$

For exercise 15, complete the tables below for non-linear functions and **graph the ordered pairs on graph paper**. For each function, connect the points and decide whether the graph should be extended.

15a)

x	$g(x)=\frac{32}{x^2}$	$(x, g(x))$
-4		
-2		
-1		
0		
1		
2		
4		

b)

x	$h(x)=2\sqrt{x-1}$	$(x, h(x))$
1		
2		
5		
10		
17		

For exercises 16 – 24, use the functions defined in exercise 15 a) and b) to find each.

16) $g(-4)$

17) $g(4)$

18) $h(17)$

19) $h(82)$

20) Find x if $g(x)=32$

21) Find x if $g(x)=2$

22) Find x if $h(x)=2$

23) $g(1)$

24) $h(1)$

25) A Metro Pass costs $25, and each ride costs $2.50. Let r = the number of rides and M = the amount remaining on the Metro Pass. Interpret each of the following.

a) $M(0)=25$

b) $M(10)=0$

1.7 SUMMARY

1. y is a $\boxed{\textbf{function}}$ of x, if for each __________ (*input* or *output*), there is only one __________ (*input* or *output*)

2. When verifying if *y is a function of x*, you must either write:

Yes, y is a function of x because for each ___ (x or y), there is only one ___ (x or y)
OR
No, y is not a function of x because for some ___ (x or y), there is more than one ___ (x or y).

In function notation, *y is a function of x* is written $y =$ __________ .

3. The ______________ of a function is the set of all allowable inputs.

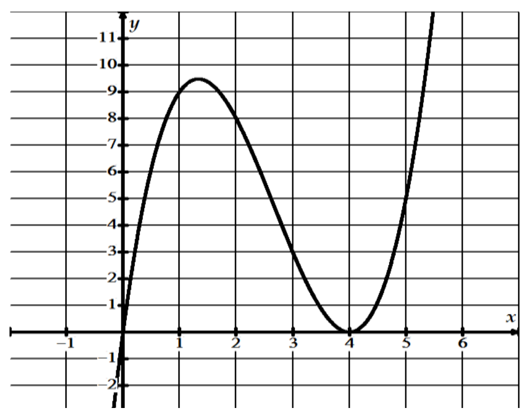
The ______________ of a function is the set of all outputs.

4.

a) The table below does not represent a function because _____________

x	y	(x, y)
10	4	(10, 4)
15	8	(15, 8)
0	−7	(0, −7)
15	−12	(15, −12)
22	20	(22, 20)
35	8	(35, 8)

b) The graph below represents the graph of a function because_____________



5. We use the ______________ ______________ test to decide if a graph is a function.

6. The cost of admission to an amusement park is a function of the number of tickets purchased. Interpret $f(2) = 125$

7. If $f(x) = x + 1$,

a) Evaluate $f(3)$

b) Evaluate $f(-1)$.

Unit 2: Equations, Inequalities, and Linear Equations

2.1 Solving Linear Equations Three Ways

Equations

An equation is a statement that two algebraic expressions are equal.

The following are examples of equations:

$$\underbrace{x+2}_{\text{This expression}} \;=\; \underbrace{-3}_{\text{This expression}} \qquad\qquad \underbrace{3(x+2)}_{\text{This expression}} = \underbrace{-3}_{\text{This expression}}$$

Notice that $x + 2$, -3, and $3(x - 2)$ are *not* equations. They are expressions. They are not equations because there is no equal sign. We have an equation when two expressions are equal.

Practice Exercises: Expression or Equation?

For exercises 1 – 5, determine whether each is an expression or an equation.

1) $3(x-1)$ ____________________________

2) $3 = x - 1$ ____________________________

3) $3x = 1$ ____________________________________

4) $3 - 2(x-1)$ _______________________________

5) $3 - 2(x-1) = 0.25$ ________________________

Solutions and Solving an Equation

The set of values that, when substituted for the variables make the equation true, are called the **solutions** of the equation.

An equation has been **solved** when all of its solutions have been found. We always verify that each solution works by substituting each solution into the original equation and determining if there is a true statement. This is known as **checking the solution**.

Example 1:
Verify that 4 is a solution to $x + 6 = 10$.

__Answer__: When $x = 4$, we substitute 4 for x in the equation $x + 6 = 10$. The result is: $4 + 6 = 10$, or $10 = 10$. This is a true statement, and verifies that 4 is a solution to $x + 6 = 10$.

Practice Exercises: Verifying Solutions

For exercises 6 – 10, verify that the given number is or is not a solution to the equation.

6) Verify that – 4 is a solution to $3(x-1)=-15$.

7) Verify that 0 is a solution to $3(x-1)=-3$.

8) Verify that – 4 is not a solution to $2(x-1)=-15$.

9) Verify that – 4 is a solution to $3(x-1)=5(x+1)$.

10) Verify that – 3 is not a solution to $3(x-1)=5(x+1)$.

Equivalent Equations

Some equations have precisely the same collection of solutions. Such equations are called equivalent equations. For example, $x-5=-1$, $x=4$, and $2x=8$ are all equivalent equations because the only solution to each is $x=4$. (Can you verify this?)

Solving Equations Symbolically

Solving One-Step Equations in One-Variable

We know that the equal sign of an equation indicates that the number represented by the expression on the left side is the same as the number represented by the expression on the right side. From this, if we can isolate the variable on one side of the equation, then we will have the solution.

To isolate the variable on one side of an equation, we use the following **properties of equations:**

1) **The <u>addition property</u> of equations states that we can obtain an equivalent equation by adding or subtracting the same number to both sides of an equation.**
2) **The <u>multiplication property</u> of equations states that we can obtain an equivalent equation by multiplying or dividing the same non-zero number on both sides of an equation.**

Example 2:

Solve the following equations: **a)** $x-1=-15$ **b)** $3x=-15$

<u>Answers</u>: **a)** To isolate *x*, we need to move – 1 to the other side of the equation by adding 1 to both sides. The result is: $-15+1=-14$. Therefore, $x=-14$ Check this solution.

b) To isolate *x*, we notice that 3 is connected to *x* by multiplication. To move 3 to the other side of the equation, we need to divide both sides of the equation by 3. Therefore, $x=-5$ Check this solution.

Solving Two-Step Equations in One-Variable

We combine the addition and multiplication properties of equations to solve more complicated equations in one variable. Our goal is to isolate the variable so that it is alone on one side.

To associate numbers and variables we use the order of operations:

1) Multiply/divide
2) Add/subtract

To solve an equation, we undo an association between numbers and letters by using the order of operations **in reverse**.

1) Add/Subtract
2) Multiply/divide

Helpful Tip: When the variable is connected to a constant and also has a numerical coefficient, **first move the constant to the other side by adding or subtracting**. Then move the numerical coefficient to the other side by multiplying or dividing. Always remember to check.

Example 3:

Solve the following equations:

a) $7x-1=-15$ **b)** $1-4x=-11$ **c)** $5-x=-1$ **d)** $3x+1=2$

Answers: **a)** We notice there are two numbers on the same side of the equation with *x*. To isolate *x*, we will follow the order of operations in reverse:

1) Move the constant, –1, to the other side of the equation by adding 1 to both sides. The result is: $7x=-14$.

2) To move the coefficient, 7, to the other side, divide both sides by 7. Therefore, $x=-2$.

3) **Check:** Replacing *x* with –2, we have

$$7(-2)-1\stackrel{?}{=}-15$$
$$-14-1\stackrel{?}{=}-15$$
$$-15=-15 \ \surd$$

b) To isolate *x*, first subtract the constant, 1, from both sides of the equation. The result is: $-4x=-12$. Then divide both sides of the equation by the coefficient, – 4. Therefore, $x=3$.

To check, replace *x* with 3:

$$1-4(3)\stackrel{?}{=}-11$$
$$1-12\stackrel{?}{=}-11$$
$$-11=-11\surd$$

c) To isolate *x*, first subtract the constant, 5, from both sides of the equation. The result is: $-x = -6$. Then divide both sides of the equation by the coefficient, –1. Therefore, $x = 6$.

To check, replace *x* with 6:

$$5-(6)\overset{?}{=}-1$$
$$-1 = -1\surd$$

d) To isolate *x*, first subtract the constant, 1, from both sides of the equation. The result is: $3x = 1$. Then divide both sides of the equation by the coefficient, 3. Therefore, $x = \frac{1}{3}$.

To check, replace *x* with $\frac{1}{3}$:

$$3\left(\frac{1}{3}\right)+1\overset{?}{=}2$$
$$1+1\overset{?}{=}2$$
$$2 = 2\ \surd$$

Approximate versus Exact Answers

If you change the fraction $\frac{1}{3}$ to a decimal, the equivalent value is a repeating, non-terminating decimal: $0.33333.....$ We call the fraction $\frac{1}{3}$ an **exact value**, and its equivalent decimal value: 0.3333… an **approximate value**. An approximate value needs to be rounded, and can be given as the answer only if the directions specify how many decimal places to round to. We will sometimes round an approximate answer in applications. For example, if the answer is money, the decimal would need to be rounded to two decimal places (hundredths place).

In algebra, the preferred value to use is the exact value, which is the fraction value. In Example 3d) above, if a calculator is used to solve $3x = 1$, the answer is 0.333333333. We know that this is a repeating, non-terminating decimal and that the calculator had to round to nine decimal places because there is no more room in the calculator window to show more 3's! When checking an equation with a rounded value, it will not always show that the left side has exactly the same result as the right side.

For example, when checking Example 3d), if *x* is replaced with 0.333333333, the result is:

$$3(0.333333333)+1 = 2$$
$$1.999999999 = 2!$$

Therefore in the last step of solving an equation, when dividing by the coefficient, if the calculator gives the result as a repeating, non-terminating decimal, simply use the calculator to change that answer to a fraction.

Practice Exercises: Solving Equations

For exercises 11 – 20, solve each equation and check the answer.

11) $x-1=-5$

12) $7+x=-15$

13) $2x-9=-15$

14) $7=15-x$

15) $5x-1=-12$

16) $-4x=24$

17) $-10=-3x$

18) $12=8-2x$

19) $7-x=-15$

20) $10+3x=-15$

Translating Words into Equations

In Unit 1, we saw that rules are equations because they have an equal sign. Now we will translate words into equations that can be solved. It will be helpful to look back at the partial list of translations in Section 1.2 of this workbook. A more complete list of translations is in your Canvas course.

Helpful Tip: As you translate words into equations, note that the verb in the sentence (such as "is") translates to the equal sign in an equation (which is a mathematical sentence.)

Practice Exercises: Translating Words into Equations

For exercises 21 – 23, write each sentence as equation with x as the input and $f(x)$ as the output.

21) The output is four less than three times the input.

22) The output is twice the input.

23) The quotient of the input and 6 is the output.

For exercises 24 – 25,

i) decide on the two variables, determine which variable is the input and which variable is the output
ii) Write a rule for each situation using function notation.

24) Find the sales tax if the tax is 7% of the cost of the item.

25) Find the value remaining on a $25 prepaid transit card if each ride costs $3.50.

Solving Equations Numerically

Now that we have discussed how to solve an equation symbolically, we will investigate how to solve an equation using a table (numerically).

To solve an equation numerically:

1) **Look in the output column for the number that stands alone on one side of the equation, replacing y or $f(x)$.**
2) **The solution to the equation is the corresponding input value(s).**

Using a Table to Solve an Equation

Use the table for Practice Exercise 21: $f(x) = 3x - 4$ **to solve the equation:** $3x - 4 = -1$.

x	Output: $f(x)$ $f(x) = 3x - 4$	$(x, f(x))$
–2	–10	(– 2, – 10)
–1	– 7	(– 1, – 7)
0	– 4	(0, – 4)
1	–1	(1, – 1)
2	2	(2, 2)
3	5	(3, 5)
4	8	(4, 8)

If we think of the equation $f(x) = 3x - 4$ in words: "The output is four less than three times the input" then we can solve equations using the table.

To solve $3x - 4 = -1$, we recognize that –1 is replacing $f(x)$. So we look in the output $(f(x))$ column for –1. When the output is –1, the input, x, is 1. Therefore the solution to this equation is $x = 1$.

Notice that no algebra was needed to solve this equation. The answer can be read directly from the table.

Helpful Tip: When solving an equation numerically, the answer must be written as "x =."

Do not give the ordered pair as the answer to an equation, since only the x-coordinate is the solution to the equation.

Example 4:

Use the table for Practice Exercise 21 to solve the following equations. Estimate where appropriate.

a) $3x - 4 = 2$ **b)** $3x - 4 = -4$ **c)** $3x - 4 = 3$

Answers:

a) When the output is 2, the input is 2. Therefore $x = 2$.

b) When the output is – 4, the input is 0. Therefore $x = 0$.

c) Since 3 is not in the output column, we recognize that 3 would be between the outputs of 2 and 5. So we look at the input values for 2 and 5, and **estimate** that when the output is 3, the input is between 2 and 3. Therefore we write the answer as: x is between 2 and 3.

Helpful Tip: You can verify the solutions to these equations by solving symbolically. To solve $3x-4=3$, add 4 to both sides and then divide by 3. The approximate solution is: $x=\frac{7}{3}\approx 2.3$, rounded to the nearest tenth. This answer agrees with the answer found using the table, which was: x is between 2 and 3.

Practice Exercises: Solving Equations Numerically

For exercises 26 and 27, solve each equation using the given table. Estimate where appropriate.

26)

x	**Output:** $f(x)$ $f(x)=10-2x$	$(x, f(x))$
–1	12	(–1, 12)
0	10	(0, 10)
1	8	(1, 8)
2	6	(2, 6)
3	4	(3, 4)
4	2	(4, 2)
5	0	(5, 0)

a) $10-2x=10$

b) $10-2x=2$

c) $10-2x=0$

d) $10-2x=7$

27)

x	**Output:** $f(x)$ $f(x)=2x-4$	$(x, f(x))$
–2	–8	(–2, –8)
–1	–6	(–1, –6)
0	–4	(0, –4)
1	–2	(1, –2)
2	0	(2, 0)

a) $2x-4=-2$

b) $2x-4=0$

c) $2x-4=-5$

Using Symbolic Techniques to Find *f(x)* when Given *x*

In Section 1.7, we used a numerical technique (the table) when evaluating expressions using function notation. For example, in exercise 27, we know from the table that if $x = 1$, then $f(x) = -2$. We also found that we can do this symbolically, that if $f(x) = 2x - 4$, then we use substitution: $f(1) = 2(1) - 4 = -2$.

Using Symbolic Techniques to Find *x* when Given *f(x)*

In Section 1.7 we found that if given the output, we can find the input in the table, that is, if $f(x) = 2x - 4$ and $f(x) = -2$, then $x = 1$.

We can now **find the input <u>symbolically</u> when given the output in function notation**: if $f(x) = -2$, then we can replace $f(x)$ with $2x - 4$. The result is the equation: $2x - 4 = -2$. We solve for x by adding 4 to both sides and dividing by 2. We now see that the solution found symbolically agrees with the table: when the output is -2, then $2x - 4 = -2$ and $x = 1$.

Example 5:

If $f(x) = 5 - 4x$, find each of the following symbolically:

a) Find $f(2)$.

b) Find $f(-2)$.

c) Find x if $f(x) = 1$.

d) Find x if $f(x) = 17$.

<u>Answers</u>:

a) If $f(x) = 5 - 4x$, then $f(2) = 5 - 4(2) = 5 - 8 = -3$.

b) If $f(x) = 5 - 4x$, then $f(-2) = 5 - 4(-2) = 5 + 8 = 13$.

c) If $f(x) = 1$, then $5 - 4x = 1$. To solve this equation, subtract 5 from both sides and divide by – 4 Therefore if $f(x) = 1$, then $x = 1$.

d) If $f(x) = 17$, then $5 - 4x = 17$. To solve this equation, subtract 5 from both sides and divide by – 4. Therefore if $f(x) = 17$, then $x = -3$.

Helpful Tips:

When given *x*, to **find** $f(x)$ **we use substitution**, as in Example 5 parts a) and b).

When given $f(x)$, **to find *x* we solve an equation**, as in Example 5 parts c) and d).

Practice Exercises: Solving Equations Symbolically when given Function Notation

For exercise 28, use $f(x) = 2x - 4$ to find the answers **symbolically**. Then check by determining if your answer agrees with the table for exercise 27.

28) a) Find $f(2)$.

b) Find $f(0)$.

c) Find x if $f(x) = -4$.

d) Find x if $f(x) = -5$.

Solving Equations Graphically

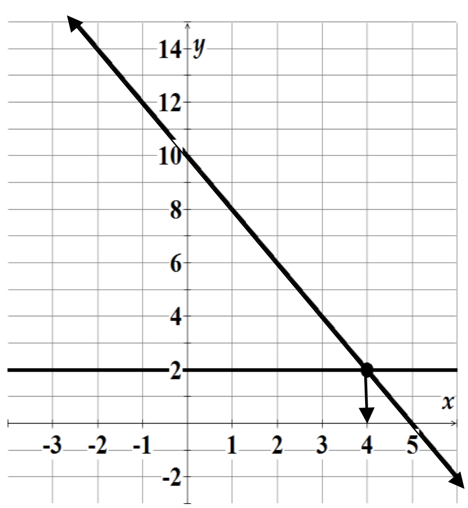
We can also use a graph to solve equations, using the following steps.

To solve an equation graphically:

1) **Look on the y-axis for the number that stands alone on one side of the equation, which replaces y or $f(x)$.**
2) **Place a straightedge parallel to the x-axis and look for points on the graph which have that y-value.**
3) **The solution to the equation is the corresponding input (x-) value(s).**

Example 6:

Use the given graph of $y = 10 - 2x$ to find the solution to the equation: $10 - 2x = 2$.

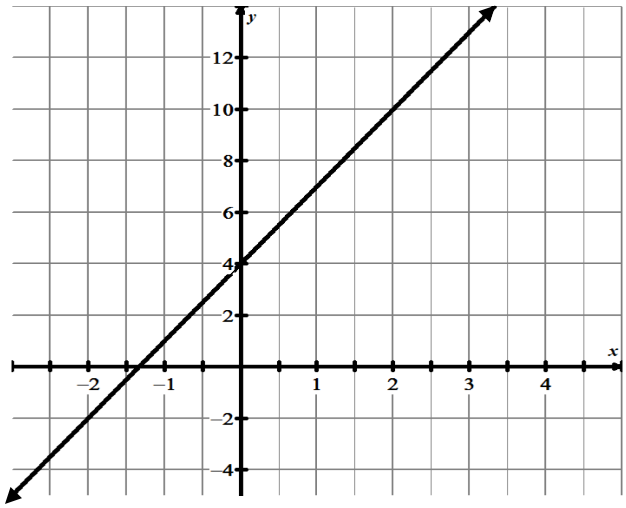


Answer:

Since the number that replaces the output, y, is 2, we look for 2 on the y-axis.

Holding a straightedge parallel to the x-axis, we see that the point on the graph in which y =2 has an x-coordinate of 4.
Therefore the solution to the equation $10 - 2x = 2$ is $x = 4$.

Example 7:
Use the given graph of $f(x) = 3x + 4$ to find the solution to each of the following equations:



a) $3x + 4 = -2$

b) $3x + 4 = 4$

c) $3x + 4 = 0$

(estimate the solution – round to the nearest tenth)

d) Find x if $f(x) = 10$.

__Answers__:

a) To solve $3x + 4 = -2$ using the graph, we find – 2 on the *y*-axis (the vertical scale is in steps of 2). Place a straightedge parallel to the *x*-axis and notice that the straightedge crosses the graph when $x = -2$. Therefore the solution to this equation is $x = -2$.

b) To solve $3x + 4 = 4$ using the graph, we find 4 on the *y*-axis. This point is directly on the *y*-axis. The graph crosses the *y*-axis when $x = 0$. Therefore the solution to this equation is $x = 0$.

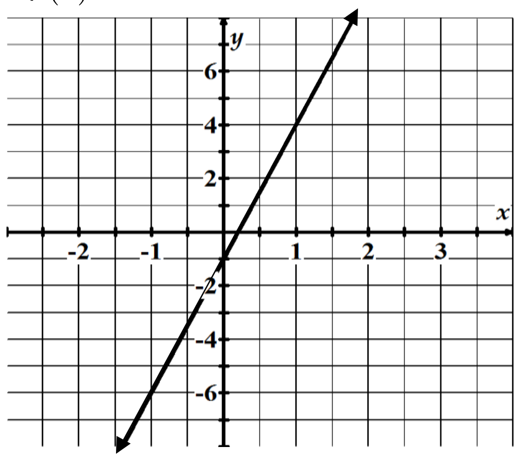
c) To solve $3x + 4 = 0$ using the graph, we find 0 on the *y*-axis. This point is directly on the *x*-axis. The graph crosses the *x*-axis when $x \approx -1.3$. Therefore we **estimate** that the solution to this equation is $x \approx -1.3$.

d) If $f(x) = 10$ then $3x + 4 = 10$. Using the graph, we find 10 on the *y*-axis. Place a straightedge parallel to the *x*-axis and notice that the straightedge crosses the graph when $x = 2$. Therefore when $f(x) = 10$, $x = 2$

Practice Exercises: Solving Equations Graphically

For exercises 29 – 30, use the given graph to solve each equation. Find approximate solutions where appropriate.

29) $f(x) = 5x - 1$



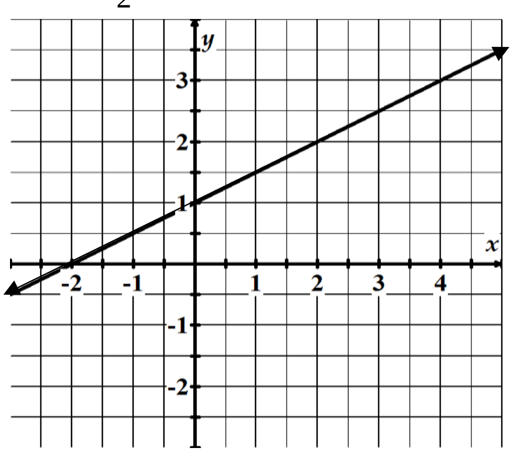
a) $5x - 1 = -6$

b) $5x - 1 = 0$
(estimate the solution)

c) $5x - 1 = 4$

d) Find x if $f(x) = -1$

30) $f(x) = \dfrac{x}{2} + 1$



a) $\dfrac{x}{2} + 1 = 0$

b) $\dfrac{x}{2} + 1 = 2$

c) $\dfrac{x}{2} + 1 = 1$

d) Find x if $f(x) = 3$.

2.1 SUMMARY

1. Fill in each blank with the word “expressions” or “equations”.

a) _____________ have equals signs.

b) _____________ do not have equals signs.

c) _____________ are solved for the value(s) of the variable.

2. To solve $2x-5=-3$ **numerically**:

Step 1: Create an input-output table for the rule $y=$____________ (or examine a given table).

Step 2: Look in the ____________ (input or output) column for -3.

Step 3: The solution is the corresponding _____________ (input or output) value.

Step 4: Write solutions in the form __ = value.

For this example the solution is ________.

x	$y=2x-5$	(x, y)
–3	–11	
–2	–9	
–1	–7	
0	–5	
1	–3	

3. To solve $2x-5=-9$ **graphically**:

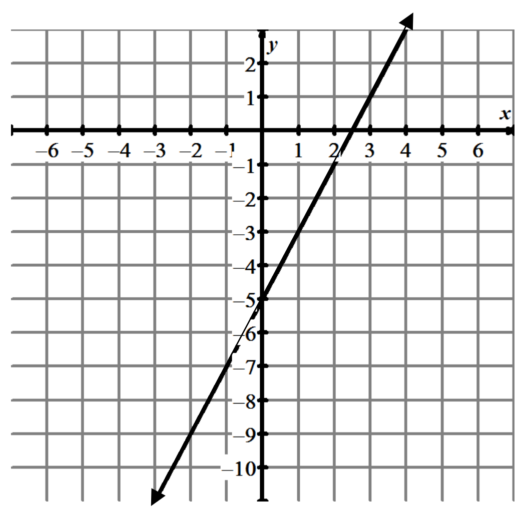
Step 1: Sketch a graph for the rule $y=$___________

Step 2: Look on the ___-axis for -9.

Step 3: The solution is the corresponding _____________ (input or output) value.

Step 4: Write solutions in the form __= value.

For this example the solution is _________.



4. To solve an equation **symbolically/algebraically**, use the addition and multiplication properties of equality, which state you can _________________ or _________________ the same number to both sides of an equation to produce an equivalent equation.

Example: Solve $2x-5=-3$ symbolically.

2.2 Solving Non-Linear Equations with Tables & Graphs; Solving Linear Equations with Parentheses

Definitions of Linear and Non-Linear Equations

Understanding the difference between linear and non-linear equations is easy if you look at the graph of the corresponding function for each. The graph of a linear function is a straight line, and the graph of a non-linear function is a curve. But how can you tell from the rule, or symbolic form, if an equation is linear or non-linear? The following is a summary of types of equations in the form you see in this course. Note: as you take more advanced courses in math, the definitions below may be expanded.

Definitions and Key Concepts		
Equations	**One-variable standard form. (The variable is x.)**	**Two-variable standard form: (The variables are x and y.)**
Linear Equation (has x)	$mx+b=0,\ m\neq 0$ For example (e.g.): $5x-1=0$	$y=mx+b,\ m\neq 0$ e.g. $y=5x-1$
Quadratic Equation (degree 2, has x^2)	$ax^2+bx+c=0,\ a\neq 0$ e.g. $2x^2+5x-1=0$	$y=ax^2+bx+c,\ a\neq 0$ e.g. $y=2x^2+5x-1$
Cubic Equation (degree 3, has x^3)	$ax^3+bx^2+cx+d=0,\ a\neq 0$ e.g. $x^3+2x^2+5x-1=0$	$y=ax^3+bx^2+cx+d,\ a\neq 0$ e.g. $y=x^3+2x^2+5x-1$
Rational Equation (fraction with variable in the denominator)	$\frac{c}{x^n}=0,\ x$ and $c\neq 0$ c is a constant and n is a positive integer. e.g. $\frac{2}{x^2}=0$	$y=\frac{c}{x^n},\ x$ and $c\neq 0$ c is a constant and n is a positive integer. e.g. $y=\frac{2}{x^2}$
Radical Function (variable is under the square root)	$\sqrt{mx+b}=0,\ m\neq 0$ e.g. $\sqrt{5x-1}=0$	$y=\sqrt{mx+b},\ m\neq 0$ e.g. $y=\sqrt{5x-1}$

Practice Exercises: Types of Linear and Non-Linear Equations

For exercises 1 – 6, identify each as a linear, quadratic, cubic, rational or radical equation.

Equation	Type of Equation	Equation	Type of Equation
1) $8=\frac{1}{x^2}$		**4)** $\frac{x}{3}-2=1$	
2) $\sqrt{2x+3}=5$		**5)** $4x^2-9=7$	
3) $x^3-27=-19$		**6)** $20-2.5x=10$	

Solving Non-Linear Equations with Tables & Graphs

In Section 2.1, we solved linear equations symbolically, numerically and graphically. However, we will only solve non-linear equations numerically and graphically.

Helpful Tip:

When solving non-linear equations numerically and graphically, there may be more than one answer. In fact, a degree 2 polynomial equation could have at most 2 solutions. A degree 3 polynomial equation could have at most 3 solutions, and so on. Be careful to find all solutions that you can read from the table or graph. You may need to estimate, as appropriate.

Example 1: Solving Non-Linear Equations Numerically

Complete the following table and use it to solve the following equations or find x.

x	y: $y=x^2-2x-3$	(x, y)
–2		
–1		
0		
1		
2		
3		
4		

a) $x^2-2x-3=0$

b) $x^2-2x-3=1$

(estimate the solution)

c) $x^2-2x-3=5$

d) Find x if $y=-3$.

<u>Answers</u>:

x	y: $y=x^2-2x-3$	(x, y)
–2	5	(–2, 5)
–1	0	(–1, 0)
0	–3	(0, – 3)
1	-4	(1,– 4)
2	–3	(2, – 3)
3	0	(3, 0)
4	5	(4, 5)

a) Notice that 0 is in the output column twice. Therefore there are two solutions to this equation. Be sure to write both.

Answer: $x = -1$ and 3.

b) Since 1 is not in the output column, we do notice that it is between 5 and 0. We write the corresponding x-values.

Answer: x is between -2 and -1, and x is between 3 and 4.

c) Answer: $x = -2$ and 4.

d) Answer: $x = 0$ and 2.

Practice Exercise: Solving Non-Linear Equations Numerically

For exercise 7, use the given table for the cubic function $f(x) = x^3 - 2x^2 - x + 2$ to solve each equation or find *x*.

7)

x	$f(x) = x^3 - 2x^2 - x + 2$	$(x, f(x))$
–2	–12	(–2, –12)
–1	0	(–1, 0)
0	2	(0, 2)
1	0	(1, 0)
2	0	(2, 0)
3	8	(3, 8)
4	30	(4, 30)

a) $x^3 - 2x^2 - x + 2 = 2$

b) $x^3 - 2x^2 - x + 2 = -12$

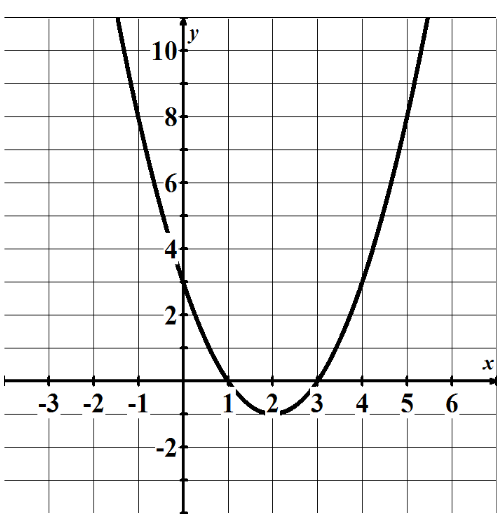
c) $x^3 - 2x^2 - x + 2 = 0$

Hint: you should have 3 solutions.

d) Find *x* if $f(x) = 8$

Example 2: Solving Non-Linear Equations Graphically

Use the given graph of $f(x) = x^2 - 4x + 3$ to solve the equations on the right or find *x*:

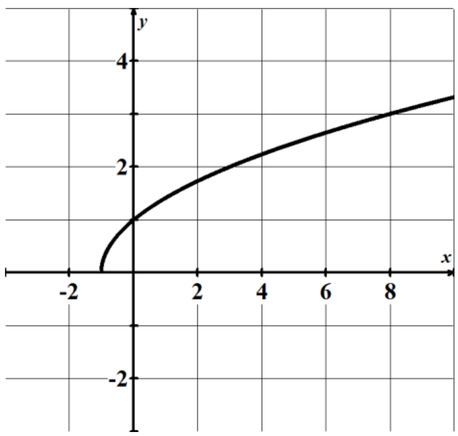


Equation:	Answers:
a) $x^2 - 4x + 3 = 8$	If we place a straightedge at $f(x) = 8$, it crosses the graph when x = −1 and 5. Therefore the solutions are x = −1 and 5.
b) $x^2 - 4x + 3 = -1$	If we place a straightedge at $f(x) = -1$, it crosses the graph when x = 2. Therefore the solution is x = 2.
c) $x^2 - 4x + 3 = 6$ (estimate the solution)	If we place a straightedge at $f(x) = 6$, we see that it does not cross the graph at exact values of x. We can estimate that the solutions are $x \approx -0.6$ and $x \approx 4.6$.
d) $x^2 - 4x + 3 = -2$	This is a quadratic equation. The graph is a parabola with a minimum at (2, –1). Therefore $f(x)$ will never be –2, and so there are **no solutions**.
e) Find x if $f(x) = -1$.	This is the same as the solution for part b): x = 2.

Practice Exercises: Solving Non-Linear Equations Graphically

For exercise 8, use the given graph of $f(x)=\sqrt{x+1}$ to solve each equation or find *x*.

8)



a) $\sqrt{x+1}=1$

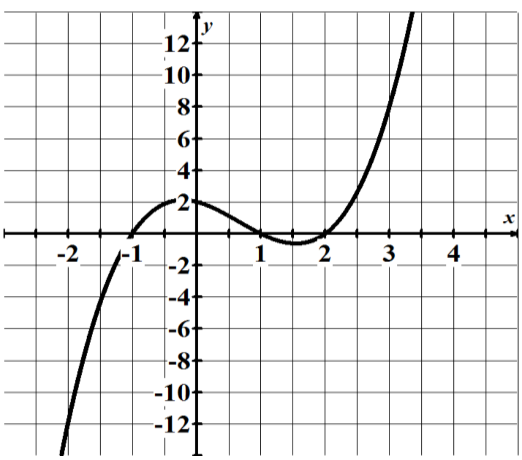
b) $\sqrt{x+1}=0$
(estimate the solution)

c) $\sqrt{x+1}=-2$

d) Find *x* if $f(x)=3$

For exercise 9, use the given graph of $f(x)=x^3-2x^2-x+2$ to solve each equation or find *x*. For parts a) and b), check your answers by looking back at how you solved the same equations using the table in Practice Exercise 7.

9)



a) $x^3-2x^2-x+2=-12$

b) $x^3-2x^2-x+2=0$
Hint: you should have 3 solutions.

c) $x^3-2x^2-x+2=4$
(estimate the solution)

d) Find *x* if $f(x)=8$

Solving Linear Equations with Parentheses Symbolically

To solve linear equations with parentheses:
1) **Use the distributive property to remove parentheses.**
2) **Combine like terms on the same side.**
3) **Move the constant term to the other side by addition or subtraction.**
4) **Move the coefficient to the other side by multiplication or division.**

Example 3:

Solve the equation $2(x+1)=18$ symbolically.

Answer:

$2(x+1)=18$

$2x+2=18$

$2x=16$

$x=8$

Explanation:

1) Use the distributive property to remove parentheses.

2) Subtract 2 from each side.

3) Divide by the coefficient 2.

4) Check the answer by replacing *x* with 8 in the original equation. To simplify the left side, we use order of operations and evaluate what is inside parentheses first.

Check:

$2(8+1)=18$

$2(9)=18$

$18=18\surd$

Example 4:

Solve the equation $5(3x-1)-6x=24-2$ symbolically.

Answer:

$5(3x-1)-6x=24-2$

$15x-5-6x=\quad 22$

$9x-5=22$

$9x=27$

$x=3$

Explanation:

1) Use the distributive property to remove parentheses.

2) Combine like terms on the same side: $15x-6x=9x$.

3) Add 5 to both sides.

4) Divide both sides by the coefficient: 9.

5) Check the answer in the original equation. Replace *x* with 3 in the original equation.

Check:

$5(3\cdot 3-1)-6(3)\overset{?}{=}24-2$

$5(9-1)-18\overset{?}{=}22$

$5(8)-18\overset{?}{=}22$

$40-18\overset{?}{=}22$

$22=22\surd$

Translating words into Equations with Parentheses

The statement: twice the sum of a number and one is eighteen, translates into the following equation: $2(x+1)=18$. Words such as: the sum of, the difference of, the product of and the quotient of are used to represent parentheses.

Helpful Tip: Be careful to translate words that represent parentheses. If this statement had been translated without parentheses, the equation and solution would have been different, and incorrect!

Practice Exercises: Solving Linear Equations with Parentheses Symbolically

For exercises 10 – 19, solve each equation and check.

10) $-2y-3(y+1)=7$

11) $-2-3(y+1)=7$

12) $2(n-6)-3n=-4+1$

13) $3-4(y-5)-3y=2$

14) $5-3(2x-4)-3(x-1)+7=0$

15) $y-3(y+1)=7$

16) $-2y-(y+1)+3=7$

17) $-2y+4(y-1)=7$

18) $6y-4(y-5)-3y=-1$

19) $8-(2x-4)-3(x+3)-18=0$

For exercises 20 – 21, translate each statement into an equation and then solve and check.

20) Four times the difference between a number and 1 is 12.

21) The sum of 5 and a number multiplied by 3 is 31.

Solving an Application by using a Rule that includes Parentheses

Example 5:

The cost of having a party at the county club is \$500 for the first 12 people, and then \$45 for each additional person.

a) Write an equation that represents the cost of this party for more than 12 people.

b) How much will the party cost if 20 people attend?

c) How many people come if the cost is \$770?

Answers:

a) To get a good understanding of this problem, it may help to make a table. The cost of the party depends on the number of people who attend, so we can define the input and output variables. Let x = number of people attending, and y = the cost of the party. We know that for up to 12 people, the price is $500. If we create a table for 13, 14, 15, 16, … people, we can better understand the process and then create a rule.

Number of people, x	**Cost of the party, y**	**Explanation**
13	545	Since there are more than 12 people, we have to pay $500. We have 1 additional person, so we pay $45 for one person. The total will be $545.
14	590	We have to pay $500. We have 2 additional people (14 – 12) so we pay an additional 2(45). The total will be $590.
15	635	We have to pay $500. We have 3 additional people (15 – 12) so we pay an additional 3(45). The total will be $635.
16	680	We have to pay $500. We have 4 additional people (16 – 12) so we pay an additional 4(45). The total will be $680.

Now that we see a process for finding the cost, we can construct a rule for the cost of more than 12 people attending the party. In words, the cost is $500 (for the first twelve people) plus 45 times the number of additional people over twelve (x – 12). Since **we have already paid for the first twelve people**, we need to subtract twelve from the number who are actually attending, x. So the symbolic form is: $y = 500 + 45(x - 12)$.

b) If 20 people attend, then x = 20. Using the rule, we have:

$y = 500 + 45(20 - 12) = 500 + 45(8) = 860$ If 20 people attend, the cost is $860.

c) If the cost is 770, then y = 770. Using the rule, we need to solve for x.

$y = 500 + 45(x - 12)$

$770 = 500 + 45x - 540$

$770 = 45x - 40$

$810 = 45x$

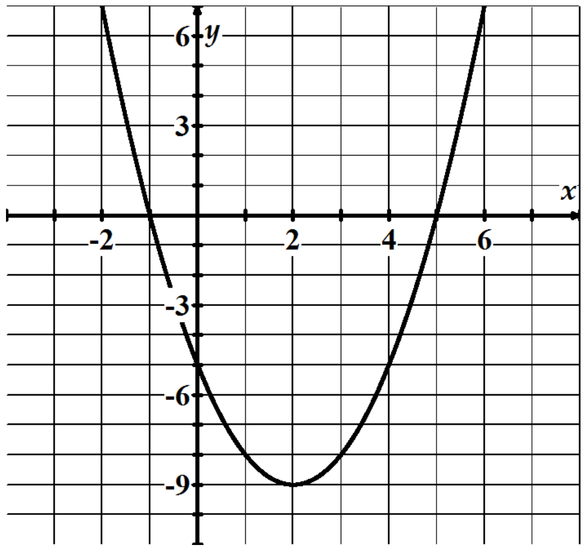
$x = 18$ people attended

1) Use the distributive property to remove parentheses.
2) Combine like terms on the same side ($500 - 540 = -40$)
3) Move the constant to the left (add 40 to both sides)
4) Divide by the coefficient, which is 45.
5) Check the solution and answer the question.

2.2 SUMMARY

1. When solving non-linear equations numerically or graphically, always look for more than one solution.

Example: Fill in the blanks using the graph. (Insert the number of solutions into each blank.)



$x^2 - 4x - 5 = -5$ has ___ solution(s). List the solution(s): ________________________

$x^2 - 4x - 5 = -9$ has ___ solution(s). List the solution(s): ______________________

$x^2 - 4x - 5 = -10$ has ___ solutions.

Example:

Use the table to solve the equations on the right.

x	$y = x^3 - 4x + 1$	(x, y)
– 3	–14	(–3, –14)
– 2	1	(–2, 1)
– 1	4	(–1, 4)
0	1	(0, 1)
1	–2	(1, –2)
2	1	(2, 1)
3	16	(3, 16)

$x^3 - 4x + 1 = 16$ has ____ solution(s).
List the solution(s):_________________________

$x^3 - 4x + 1 = 1$ has ___ solution(s).
List the solution(s):_________________________

The solution to the equation $x^3 - 4x + 1 = -10$ would fall between the numbers _____ and _____.

2. To solve an equation containing parentheses, the ________________________ property must be applied to remove them.

Example: Circle the correct solution to the equation: $8 - 3(x+1) = 15$. Show the check.

Solution 1:

$8 - 3(x+1) = 15$

$5(x+1) = 15$

$5x + 5 = 15$

$5x = 10$

$x = 2$

Solution 2:

$8 - 3(x+1) = 15$

$8 - 3x - 3 = 15$

$-3x + 5 = 15$

$-3x = 10$

$x = -\dfrac{10}{3}$

2.3 Solving Linear Equations with Variable Terms on Both Sides Symbolically

In Section 2.2, we discussed how to solve an equation that has parentheses by using the distributive property and combining like terms. In this section, we will solve equations that have a variable term on both sides.

Example 1:

Solve: $2x+3=5x-9$

Answer:

$$\begin{array}{rcl} 2x+3 &=& 5x-9 \\ -5x & & -5x \\ \hline -3x+3 &=& -9 \\ -3 & & -3 \\ \hline -3x &=& -12 \\ x &=& 4 \end{array}$$

Check:

$$2(4)+3 \stackrel{?}{=} 5(4)-9$$

$$8+3 \stackrel{?}{=} 20-9$$

$$11 = 11\checkmark$$

Explanation:

1) Since our goal is to isolate the variable, it is important to have the variable term on one side. So we move $5x$ to the left side by subtracting $5x$ from both sides.

2) Next we move the constant term, 3, to the right side by subtracting 3 from both sides.

3) Finally, we divide both sides by the coefficient, which is -3.

4) Remember to check the answer by replacing x with 4 in the original equation.

Practice Exercises: Solving Equations with a Variable Term on Both Sides

For Exercises 1 – 3, solve each equation and check.

1) $6x-4=2x+8$

2) $8a-5=3a-5$

3) $5x+14=6x-12$

Example 2:

Solve: $2(x+3)=5(x-9)$

<u>Answer:</u>

$$2(x+3)=5(x-9)$$
$$2x+6=5x-45$$
$$-3x+6=-45$$
$$-3x=-51$$
$$x=17$$

Check:

$$2(17+3)\overset{?}{=}5(17-9)$$
$$2(20)\overset{?}{=}5(8)$$
$$40=40\ \surd$$

Explanation:

1) Use the distributive property to remove parentheses.
2) Move the variable term to the left side by subtracting $5x$ from both sides. $(2x-5x=-3x)$
3) Move the constant, 6, to the right side by subtracting 6 from both sides. $(-45-6=-51)$
4) Divide both sides by the coefficient, -3.
5) Check the answer by replacing x with 17 in the original equation.

Practice Exercises: Solving Equations with Parentheses and a Variable Term on Both Sides

For exercises 4 – 9, solve each equation and check.

4) $3(x+3)=2(x-9)$

5) $2(x+3)=3(x-9)$

6) $x+3=2(x-9)$

7) $-(x+3)=2(x-9)$

8) $5(x-3)=3(x-5)$

9) $3(x+3)=-2(x-9)$

Solving an Equation by Combining Like Terms

Example 3: Solve $2y-7-4y=3+3y$

<u>Answer</u>:

$$2y-7-4y=3+3y$$
$$-2y-7=3+3y$$
$$-5y-7=3$$
$$-5y=10$$
$$y=-2$$

Check:

$$2y-7-4y=3+3y$$
$$2(-2)-7-4(-2)\overset{?}{=}3+3(-2)$$
$$-4-7+8\overset{?}{=}3-6$$
$$-3=-3\surd$$

Explanation:

1) Since there are no parentheses to remove, we combine like terms on the same side. On the left, $2y-4y=-2y$.
2) Move the variable term to the left by subtracting $3y$ from both sides. On the left, $-2y-3y=-5y$.
3) Move the constant to the right by adding 7 to both sides: 7 + 3 = 10.
4) Divide by the coefficient: – 5.
5) Check by replacing y with – 2 in the original equation.

Solving an Equation by Eliminating Parentheses and Combining Like Terms

Example 4:

Solve: $5(x-1)-3x = 4-x$

Answer:

$$5(x-1)-3x = 4-x$$
$$5x-5-3x = 4-x$$
$$2x-5 = 4-x$$
$$3x-5 = 4$$
$$3x = 9$$
$$x = 3$$

Check:

$$5(x-1)-3x = 4-x$$
$$5(3-1)-3(3) \stackrel{?}{=} 4-3$$
$$5(2)-9 \stackrel{?}{=} 1$$
$$1 = 1\checkmark$$

Explanation:

1. **1)** Use the distributive property to remove parentheses by distributing the 5.
2. **2)** Combine like terms on the same side. On the left side, $5x-3x = 2x$.
3. **3)** Move the variable term to the left side by adding x to both sides.
4. **4)** Move the constant to the right by adding 5 to both sides.
5. **5)** Divide by the coefficient, 3.
6. **6)** Check by replacing x with 3 in the original equation.

Example 5:

Solve: $2y+3 = 8y-3(y+1)$

Answer:

$$2y+3 = 8y-3(y+1)$$
$$2y+3 = 8y-3y-3$$
$$2y+3 = 5y-3$$
$$-3y = -6$$
$$y = 2$$

Check:

$$2y+3 = 8y-3(y+1)$$
$$2(2)+3 \stackrel{?}{=} 8(2)-3(2+1)$$
$$4+3 \stackrel{?}{=} 16-3(3)$$
$$7 \stackrel{?}{=} 16-9$$
$$7 = 7\checkmark$$

Explanation:

1. **1)** Use the distributive property to remove parentheses by distributing the -3.
2. **2)** Combine like terms on the same side. On the right side, $8y-3y = 5y$.
3. **3)** Move the variable term to the left side by subtracting $5y$ from both sides.
4. **4)** Move the constant to the right side by subtracting 3 from both sides.
5. **5)** Divide by the coefficient, –3.
6. **6)** Check by replacing y with 2 in the original equation.

Summary: How to Solve Linear Equations with Variables on Both Sides

To solve linear equations with the variable on both sides, simplify each side.

1) **Use the distributive property to remove parenthesis.**
2) **Combine like terms on the same side.**
3) **Move the variable terms to the same side (often the left side) by addition or subtraction.**
4) **Move the constant term to the other side by addition or subtraction.**
5) **Move the coefficient to the other side by multiplication or division.**

Notice that there is now one additional step in the summary box that you had seen in the previous section.

Practice Exercises: Solving Linear Equations

For exercises 10 – 19, solve each equation and check.

10) $5x-3=2x-9$

11) $x+4=5(x-8)$

12) $2(y-7)=3(1+y)-8$

13) $x-6-3x=4(1-x)$

14) $3(y+3)=8y-3(y+2)$

15) $3x+7=-3-(x+2)$

16) $-4x-(-4-3x)=-3x-2x-(3-6x)+1$

17) $5-2(2y+3)=-3(y+1)$

18) $5(x+3)=8x-6(x+1)$

19) $6-(x+3)=8-2(x-1)$

Solving Applications by using Equations

Example 6:

Gabriella has been offered a job selling shoes, but before she accepts the job she needs to decide how she wants to be paid each week. If she selects payment plan A, she will get paid $550 plus 10% of sales. If she selects payment plan B, she will get paid $600 plus 5% of sales.

a) If Gabriella sells $800 in shoes, how much will she get paid if she selects plan A?

b) Let x represent the amount of sales she makes each week. Write an equation that represents her salary, y, if she selects payment plan A.

c) If Gabriella sells $800 in shoes, how much will she get paid if she selects plan B?

d) Let x represent the amount of sales she makes each week. Write an equation that represents her salary, y, if she selects payment plan B.

e) If Gabriella sells $1200 in shoes, how much will she get paid if she selects plan A?

f) If Gabriella sells $1200 in shoes, how much will she get paid if she selects plan B?

g) For what amount of sales will her salary from plan A and plan B be the same?

h) What advice would you give Gabriella to help her decide whether to select payment plan A or plan B?

<u>Answers</u>:

a) In plan A, Gabriella gets $550 plus 10% of sales. If her sales are $800, this translates to the equation: salary = 550 + 0.10(800) = $630.

b) The equation for payment plan A is: $y = 550 + 0.1\,x$.

c) In plan B, Gabriella gets $600 plus 5% of sales. If her sales are $800, this translates to the equation: salary = 600 + 0.05(800) = $640.

d) The equation for payment plan B is: $y = 600 + 0.05\,x$.

e) If her sales are $1200 and she selects plan A, her salary = 550 + 0.10(1200) = $670.

f) If her sales are $1200 and she selects plan B, her salary = 600 + 0.05(1200) = $660.

g) Gabriella's salary will be the same when her plan A salary = her plan B salary. In symbolic form, this happens when: $550 + 0.1x = 600 + 0.05x$. Solving this equation, we move the variable term to the left by subtracting $0.05x$ from both sides. We move the constant to the right by subtracting 550 from both sides. The result is: $0.05\,x = 50$. Divide both sides by the coefficient, 0.05, and we have $x = 1000$. When Gabriella sells $1000 in shoes, her salary is the same whether she selects plan A or plan B.

h) If Gabriella sells less than $1000 in shoes, then she should select plan B. However, if she expects to sell more than $1000 in shoes each week, then she should select plan A.

2.3 SUMMARY

1. **Steps for Solving Linear Equations Symbolically**

- Apply the ________________ property to expressions with parentheses.
- ______________ like terms.
- Get all the terms with the variable to be solved for onto one side of the equation and constant terms to the other side of the equation.
- Add like terms yet again.
- Solve for the variable using the multiplication property of equations.

2. When solving an equation with variables on both sides, ______________ the expressions on each side before moving variables to one side and constants to other.

Example: Describe the steps to solve the equation: $85+5(x-11)=100-2x$.

$85+5(x-11)=100-2x$ -- Given problem.

$85+5x-55=100-2x$ --____________________________________

$5x+30=100-2x$ --____________________________________

$7x+30=100$ --____________________________________

$7x=70$ --____________________________________

$x=10$ --____________________________________

Check this solution:

$$85+5(x-11)=100-2x$$

$$85+5(\_\_\_-11)\stackrel{?}{=}100-2(\_\_\_)$$

$$\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_ \surd$$

2.4 Solving Linear Equations with Fractions and Solving Proportions

Solving Linear Equations with Fractions by using Steps

To solve a linear equation with fractions, we can use the steps listed in the previous section. When solving an equation with fractions, it is helpful to use a calculator.

Example 1:

Solve the equation: $\frac{2}{3}x = 9$.

Answer:

To isolate x in the equation $\frac{2}{3}x = 9$ we notice that x is connected to $\frac{2}{3}$ by multiplication. So we need to divide both sides of the equation by $\frac{2}{3}$. On the right side, $9 \div \frac{2}{3} = 9\left(\frac{3}{2}\right) = \frac{27}{2} = 13.5$.

If you use a calculator, press $9 \div \frac{2}{3}$ and press "enter". Therefore, $x = \frac{27}{2} = 13.5$.

Either the fraction or decimal answer can be given, since both answers are exact.

$$Check: \ \frac{2}{3}\left(\frac{27}{2}\right) = 9 \ \surd$$

Example 2:

Solve the equation: $\frac{x}{3} + \frac{1}{6} = 9$.

Answer:

Using the steps for solving linear equations that we have seen previously, first subtract the constant, $\frac{1}{6}$, from both sides: $9 - \frac{1}{6} = \frac{53}{6}$.

Then to isolate x, multiply both sides by 3: $\frac{53}{6}(3) = \frac{53}{2} = 26.5$. Therefore, $x = \frac{53}{2} = 26.5$.

Either the fraction or decimal answer can be given, since both answers are exact.

$$Check: \ \frac{(x)}{3} + \frac{1}{6} = \frac{(26.5)}{3} + \frac{1}{6} = \frac{53.0}{6} + \frac{1}{6} = \frac{54}{6} = 9 \ \surd$$

Example 3:

Solve the equation: $\frac{2}{3}x = x + 3$.

Answer:

First we need to get both variable terms on the same side. Remember that x is the same as $1x$.

$\frac{2}{3}x = x + 3$ subtract $1x$ from both sides

$-\frac{1}{3}x = 3$ divide both sides by $-\frac{1}{3}$

$x = -9$

Check: replacing x with -9,

$\frac{2}{3}(-9) \stackrel{?}{=} -9 + 3$

$-6 = -6\surd$

Solving Linear Equations with Fractions by Clearing Denominators

It seems most reasonable that an equation without any fractions would be easier to solve than an equation with fractions. Our goal, then, is to convert any equation with fractions to an equation that contains no fractions. The process of eliminating fractions from equations is called "Clearing Denominators." This is easily done by making an equivalent equation using the multiplication property of equations.

In Section 2.1, we learned that equivalent equations have precisely the same collection of solutions. If we are given an equation, we can obtain an equivalent equation by applying properties of equations: we can add, subtract, multiply or divide both sides of the equation by the same number.

For example, to isolate x in the equation $\frac{x}{3} = 9$, we would multiply both sides of the equation by 3. Therefore $\frac{x}{3} = 9$ and $x = 27$ are equivalent equations.

To solve equations with fractions by clearing denominators, we will use the multiplication property of equations.

To develop this method, let's consider the equation: $\frac{1}{3} + \frac{x}{4} = \frac{17}{6}$.

To clear the denominators, we need to find a number which can be divided by 3, 4, and 6. Since the least common denominator (LCD) of those three numbers is 12, we will multiply both sides by 12: $12\left(\frac{1}{3} + \frac{x}{4}\right) = 12\left(\frac{17}{6}\right)$

When we apply the distributive property, we recognize that we are multiplying every term on both sides by the LCD.

$$12\left(\frac{1}{3}\right)+12\left(\frac{x}{4}\right)=12\left(\frac{17}{6}\right)$$

The result is: $4+3x=34$. Now there are no more fractions, and we can solve this equation using our previous steps to obtain the answer, which is $x=10$.

To Solve an Equation with Fractions by Clearing Denominators

To clear denominators in an equation with fractions, multiply every term on both sides of the equation by the LCD of the denominators.

In Examples 4 – 6, we return to Examples 1 – 3 in this section, and solve each one by clearing denominators.

Example 4:

Solve the equation: $\frac{2}{3}x=9$ by clearing denominators.

Answer:

There is only one denominator in this equation, 3. So we multiply **every term** on both sides by 3.

$$3\left(\frac{2}{3}x\right)=3(9)$$

The resulting equation is $2x=27$

$$x=13.5$$

See Example 1 for the check.

Example 5:

Solve the equation: $\frac{x}{3}+\frac{1}{6}=9$ by clearing denominators.

Answer:

The LCD in this equation is 6. So we multiply **every term** on both sides by 6.

$$6\left(\frac{x}{3}\right)+6\left(\frac{1}{6}\right)=6(9)$$

$$2x+1=54$$

$$2x=53$$

$$x=\frac{53}{2}=26.5$$

Example 6:

Solve the equation: $\frac{2}{3}x = x+3$ by clearing denominators.

Answer:

There is only one denominator in this equation, 3. So we multiply **every term** on both sides by 3.

$$3\left(\frac{2}{3}\right)x = 3(x)+3(3)$$
$$2x = 3x+9$$
$$-1x = 9$$
$$x = -9$$

Three Ways to Solve an Equation with Fractions and Parentheses

Next we look at how to solve an equation with fractions and parentheses. Although three different ways are presented in Example 7, you will only need to solve by clearing denominators (method iii) if the directions specify using that technique.

Example 7:

Solve the equation: $\frac{2}{7}(x-14) = 8$.

Answer:

i) If you are using a calculator, then it may be easy to use the distributive property.

$\frac{2}{7}(x-14) = 8$	distribute $\frac{2}{7}$	***Check:*** replacing ***x*** with 42,
$\frac{2}{7}x-4 = 8$	add 4 to both sides	$\frac{2}{7}(42-14) \overset{?}{=} 10$
$\frac{2}{7}x = 12$	divide both sides by $\frac{2}{7}$	$\frac{2}{7}(28) \overset{?}{=} 8$
$x = 42$		$8 = 8\surd$

ii) Another way to solve the equation $\frac{2}{7}(x-14) = 8$, is to first divide both sides by $\frac{2}{7}$.

$$\left[\cancel{\frac{2}{7}}(x-14)\right] \div \cancel{\frac{2}{7}} = 8 \div \frac{2}{7}$$

The result is: $x-14 = 28$

$x = 42$

iii) Finally, we can solve this equation by clearing denominators.

$7\left[\frac{2}{7}(x-14)\right]=7[8]$ multiply both sides of the equation by the LCD, 7

$2(x-14)=56$ next we distribute the 2

$2x-28=56$ add 28 to both sides and divide by 2

$x=42$

Helpful Tip: Notice that when multiplying $7\left[\frac{2}{7}(x-14)\right]$ you only need to multiply $7\left[\frac{2}{7}\right]$. When multiplying each side by the LCD, do not multiply any terms that are inside parentheses.

Practice Exercises: Solving Linear Equations with Fractions

For exercises 1 – 8, solve each equation and check.

1) $\frac{2}{7}x-9=-17$

2) $-9-\frac{2}{3}x=-17$

3) $\frac{1}{8}x-\frac{2}{3}=\frac{1}{12}$

4) $\frac{2}{7}x-8=\frac{5}{7}x-17$

5) $\frac{2}{3}x+2=\frac{3}{5}x-7$

6) $\frac{5}{8}x-2=\frac{7}{12}x-1$

7) $\frac{3}{4}(x+4)=x-2$

8) $\frac{1}{2}(3x-5)=\frac{2}{3}x+1$

Solving Proportions

Ratios

Ratios are used to compare amounts or describe a relationship between two quantities, usually with the same units. In algebra, we work with ratios that have the form $\frac{a}{b}$ where $b \neq 0$.

Example 8:

A rectangle is 6 inches long and 10 inches wide. What is the ratio of length to width?

Answer:

The ratio of length to width is $\frac{6\ inches}{10\ inches}$. The units are the same (inches) and so they cancel. This ratio can be simplified to $\frac{3}{5}$.

Proportions

A **proportion** is an equation which states that two ratios are equal. If you know one ratio in a proportion, you can use that information to find values in the other equivalent ratio.

Cross Multiplication Property of Proportions

In Example 8 it was stated that $\frac{6}{10} = \frac{3}{5}$. **In a proportion, cross-products are equal.** We call this cross multiplying, because the pattern of multiplication looks like an "X" or a crisscross. For the proportion $\frac{6}{10} = \frac{3}{5}$, the cross products are found using the following pattern: $\frac{6}{10} = \frac{3}{5}$.
In this proportion, if you multiply 3 • 10 = 30 and multiply 5 • 6 = 30, then both products are equal.

Solving a Proportion

If you know that the relationship between quantities is proportional, you can use the cross-multiplication property of proportions to find missing quantities.

To Solve a Proportion:
1) **Multiply cross-products.**
2) **Solve the resulting equation for *x*.**
3) **Check the answer in the original proportion.**

Example 9:

Solve the proportions: **a)** $\frac{6}{90} = \frac{x}{15}$ **b)** $\frac{x}{10} = \frac{7}{35}$

<u>Answers:</u>

a)
$$\frac{6}{90} = \frac{x}{15}$$
$$90x = 6(15)$$
$$90x = 90$$
$$x = 1$$

Check:
$$\frac{6}{90} \stackrel{?}{=} \frac{1}{15}$$
$$90(1) \stackrel{?}{=} 6(15)$$
$$90 = 90 \ \surd$$

b)
$$\frac{x}{10} = \frac{7}{35}$$
$$35x = 7(10)$$
$$35x = 70$$
$$x = 2$$

Check:
$$\frac{2}{10} \stackrel{?}{=} \frac{7}{35}$$
$$2(35) \stackrel{?}{=} 7(10)$$
$$70 = 70 \ \surd$$

Cross-products that Require Parentheses

If a numerator and/or denominator of the ratios in a proportion has **more than one term**, you will need to use parentheses when writing the cross-products.

Example 10:

Solve the proportions: **a)** $\frac{x-2}{9}=\frac{x}{15}$ **b)** $\frac{x-1}{10}=\frac{x+3}{5}$

Answers:

a)

$\frac{x-2}{9}=\frac{x}{15}$

$9x=15(x-2)$ use parenthesis

$9x=15x-30$ apply the distributive property

$-6x=-30$ subtract $15x$ from both sides

$x=5$ divide by the coefficient -6

Check:

$\frac{5-2}{9}\stackrel{?}{=}\frac{5}{15}$

$\frac{3}{9}\stackrel{?}{=}\frac{5}{15}$

$3(15)\stackrel{?}{=}9(5)$

$45=45\surd$

b)

$\frac{x-1}{10}=\frac{x+3}{5}$

$5(x-1)=10(x+3)$ use parenthesis

$5x-5=10x+30$ apply the distributive property

$-5x-5=30$ subtract $10x$ from both sides

$-5x=35$ add 5 to both sides

$x=-7$ divide by the coefficient, -5

Check:

$\frac{-7-1}{10}\stackrel{?}{=}\frac{-7+3}{5}$

$\frac{-8}{10}\stackrel{?}{=}\frac{-4}{5}$

$-4(10)\stackrel{?}{=}-8(5)$

$-40=-40\surd$

Recognizing the Difference between a Proportion and an Equation with Fractions

Cross-multiplication can only be used when solving a proportion, so it is important to recognize when an equation is a proportion.

A proportion is two equal ratios (fractions). Therefore, a proportion will take the following form: $\frac{\square}{\square}=\frac{\square}{\square}$

In this course, the variable in a proportion will be in the numerator. If there is more than one term in the numerator, place parentheses around the expression and be sure to use the distributive property to remove parentheses.

Example 11:

Solve each of the following equations: **a)** $\frac{3x}{4} = 5 - \frac{3x}{6}$ **b)** $\frac{3x}{4} = \frac{x-5}{8}$

Answers:

a) This equation is not a proportion, so we can solve by clearing denominators. We will multiply every term on each side by the LCD, 12.

$$12\left(\frac{3x}{4}\right) = 12(5) - 12\left(\frac{3x}{6}\right)$$

$9x = 60 - 6x$ add $6x$ to each side

$15x = 60$

$x = 4$

Check:

$$\frac{3(4)}{4} \stackrel{?}{=} 5 - \frac{3(4)}{6}$$

$$\frac{12}{4} \stackrel{?}{=} 5 - \frac{3(4)}{6}$$

$$3 \stackrel{?}{=} 5 - 2$$

$$3 = 3 \surd$$

b) This equation is a proportion, so we can solve by multiplying cross products. Remember to use parentheses.

$\frac{3x}{4} = \frac{x-5}{8}$ use parenthesis

$8(3x) = 4(x-5)$ apply the distributive property

$24x = 4x - 20$ subtract $4x$ from each side

$20x = -20$

$x = -1$

Check:

$$\frac{3(-1)}{4} \stackrel{?}{=} \frac{(-1)-5}{8}$$

$$\frac{-3}{4} \stackrel{?}{=} \frac{-6}{8}$$

$$-3(8) \stackrel{?}{=} 4(-6)$$

$$-24 = -24 \surd$$

Practice Exercises: Solving Linear Equations with Fractions and Proportions

For exercises 9 – 12, solve each equation and check.

9) $\frac{n}{5} = \frac{10}{8}$

10) $\frac{x-1}{45} = \frac{7}{9}$

11) $\frac{x-2}{6} = \frac{x+3}{16}$

12) $\frac{2}{3}x + 4 = x + 1$

2.4 SUMMARY

1. STEPS for Solving Linear Equations Symbolically

- Clear denominators, if applicable.
- Apply the ________________ property to expressions with parentheses.
- ______________ like terms.
- Get all the terms with the variable to be solved for onto one side of the equation and constant terms to the other side of the equation.
- Add like terms yet again.
- Solve for the variable using the multiplication property of equations.

Example:

To clear denominators, every term on both sides should be multiplied by the ________________

__________________ ________________________.

a) For $\frac{1}{3}x+4=7$ multiply each term on both sides by _____ .

b) For $5-\frac{2}{5}x=\frac{1}{3}$ multiply each term on both sides by _____ .

c) For $\frac{7}{2}=\frac{1}{2}x-\frac{1}{6}$ multiply each term on both sides by _____ .

2. Cross-multiplication can be used to solve a ________________________

$$\frac{a}{b}=\frac{c}{d} \quad \rightarrow \quad \_\_\_\_\_ = \_\_\_\_\_ \quad \textit{where } b \neq \_\_\_ \quad \textit{and } d \neq \_\_\_\_$$

Examples: Decide which equation is a proportion or not a proportion, and complete the solutions.

This equation is a ______________________
proportion or not a proportion

versus

This equation is a ______________________
proportion or not a proportion

Denominators are cleared here by cross-multiplication. *Note the need for parentheses.

$$\frac{5x-1}{9}=\frac{x+7}{3}$$

$$3(5x-1) = 9(x+7)$$

$$\frac{5}{9}x-1=\frac{x}{3}+7$$

$$9\left(\frac{5}{9}x-1\right)=9\left(\frac{x}{3}+7\right)$$

Cross-multiplication cannot be applied. To clear denominators, multiply **every term** by the LCD, 9.

2.5 Inequalities, Line Graphs and Solving Linear Inequalities in One Variable

Relationships of Inequality

We have discovered that an equation is a mathematical way of expressing the relationship of equality between quantities. Not all relationships need be relationships of equality, however. Certainly the number of human beings on earth is greater than 20. Also, the National Institutes of Health recommend that the average intakes of Vitamin C for children and adolescents aged 1-18 years range from 75.6 mg/day to 100 mg/day. These types of relationships are not relationships of equality, but rather, relationships of **inequality**.

Linear Inequalities

A **linear inequality** is a mathematical statement that one linear expression is greater than or is less than another linear expression. In an equation, the verb is =. In an inequality, the verb is one of the four symbols below.

Inequality Notation

The following notation is used to express relationships of inequality:

$<$	**is less than**
$>$	**is greater than**
$\leq$	**is less than or equal to**
$\geq$	**is greater than or equal to**

What are some of the differences between a Linear Equation and a Linear Inequality?

In a linear equation, we have seen that there is one solution. When we follow the steps to solve a linear equation, we are isolating x and the last step of the solution is x = a constant.

Note that the linear inequality $x > 12$ has infinitely many solutions. Any number strictly greater than 12 will satisfy the statement. Some solutions are 13, 15, 90, 12.1, 16.3 and 102.51. We call the set of numbers that satisfy an inequality the **solution set**. To help visualize the solution set, we use a line graph.

Line Graphs

A line graph is a number line that provides a graphical representation of the solution set of an inequality.

To construct a line graph:

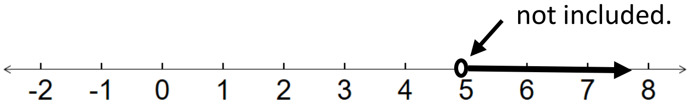
1) **Draw a number line. Label tick marks with the number in the inequality and several integers that are less than and several integers that are greater than the number. We often include 0 on a line graph, when appropriate.**
2) **Above the given number, use an open circle on the number line for $<$ or $>$ inequalities. The inequalities $\leq$ or $\geq$ include the number in the solution set, so we use a closed circle on the number line.**
3) **In $<$ or $\leq$ inequalities, shade the numbers to the left of (less than) the given number and draw an arrow in the direction of $-\infty$ (negative infinity).**
   In $>$ or $\geq$ inequalities, shade the numbers to the right of (greater than) the given number and draw an arrow in the direction of ∞ (positive infinity).

Example 1:

Draw a line graph for the inequality: $x > 5$.

Answer:

Note that an open circle indicates that the point is not included.

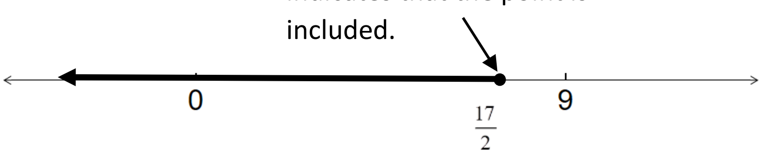


Example 2:

Draw a line graph for the inequality: $x \leq \frac{17}{2}$.

Answer:

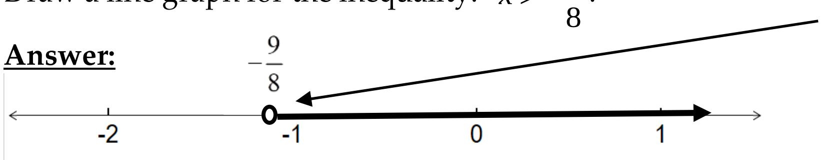
Note that a closed circle indicates that the point is included.



Example 3:

Draw a line graph for the inequality: $x > -\frac{9}{8}$.

Answer:



Note the open circle on the number line, showing that the numbers satisfying this inequality are strictly greater than (and not including) $-\frac{9}{8}$.

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Compound Inequalities

At the beginning of this section, it was stated that the average intakes of Vitamin C for children and adolescents aged 1-18 years range from 75.6 mg/day to 100 mg/day. In words, this means that the average intake is greater than or equal to 75.6 mg and less than or equal to 100 mg. In symbols, we write this as $x \geq 75.6$ and $x \leq 100$, meaning that the average intake is between 75.6 and 100, and includes both 75.6 and 100.
When the solution set represents numbers "between" and/or including two numbers, we combine the two inequalities in one statement. This is a **compound inequality**.

"Between" Notation

The format for a compound inequality using between notation is:

Smaller number	$<$ or $\leq$	x	$<$ or $\leq$	**Larger number**

In the previous example, the numbers between 75.6 and 100 including 75.6 and 100, can be written as $75.6 \leq x \leq 100$.

Helpful Tips:

1) When writing a compound inequality, only use $<$ or $\leq$.
2) Although we can think of the numbers between 75.6 and 100 as the two inequalities, $x \geq 75.6$ and $x \leq 100$, when we write this as the compound inequality $75.6 \leq x \leq 100$, we read it as "**the numbers between 75.6 and 100, inclusive.**" When we are including **both values**, we can simply say "inclusive" rather than "including **both** 75.6 and 100."

Example 4:

Write the following pair of inequalities as a compound inequality using the correct notation: $x < 0$ and $x > -3$. Then graph the solution set on a real number line.

Answer:

The numbers that are less than 0 and greater than -3 are between -3 and 0. Using "between" notation, we write this compound inequality as: $-3 < x < 0$.

The graph of the solution set is:

To construct a line graph for a Compound Inequality:

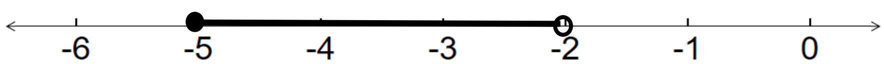
1) **Draw a number line. Label tick marks with the numbers in the inequality, being sure that the smaller number is on the left. Add tick marks for the integers between. We often include 0 on a line graph, when appropriate.**
2) **Above the given number, use an open circle on the number line for < or > inequalities. The inequalities $\leq$ or $\geq$ include the number in the solution set, so we use a closed circle on the number line.**
3) **Shade the number line values between the two given numbers.**

Example 5:

Write the inequalities $x \geq -5$ and $x < -2$ as a compound inequality, and draw a line graph for the compound inequality.

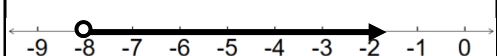
Answer:

The numbers that are greater than or equal to – 5 and less than – 2 are between – 5 and – 2, including – 5. Using "between" notation, we write this compound inequality as: $-5 \leq x < -2$. The line graph for the compound inequality is:



Practice Exercises: Inequalities and Line Graphs

For exercises 1 – 7, fill in the missing graphical, symbolic, and verbal representations.

	NUMBER LINE	SYMBOLS	WORDS
1)			x is less than or equal to – 8
2)			
3)		$-4 \leq x \leq 1$	
4)		$x > 12$	
5)			
6)			x is between – 2 and 3, including 3.
7)		$x < 0$	

Solving Linear Inequalities in One Variable

Inequalities can be solved by basically the same methods as linear equations, but with one important exception, as we shall see.

The Algebra of Linear Inequalities

If a, b, and c are real numbers, we have the following properties of inequalities:

Addition/Subtraction Property of Inequalities:

If any real number, c, is added or subtracted from both sides of an inequality, the direction of the inequality sign remains the same.

If $a<b$, then $a+c<b+c$

Multiplication/Division by a Positive Number Property

If both sides of an inequality are multiplied or divided by the same positive number, the direction of the inequality sign remains the same.

If $c>0$ and $a<b$, then $ac<bc$

We can verify the addition/subtraction property and the multiplication/division property of inequalities.

Consider the inequality: 2 < 6.

If we add 5 to both sides, we have:
$$\begin{aligned}2+5&<6+5\\ 7&<11 \quad \text{which is true}\end{aligned}$$

If we subtract 5 from both sides, we have:
$$\begin{aligned}2-5&<6-5\\ -3&<1 \quad \text{which is true}\end{aligned}$$

If we multiply both sides of the inequality by 5, a positive number, we have:
$$\begin{aligned}5(2)&<5(6)\\ 10&<30 \quad \text{which is true}\end{aligned}$$

If we divide both sides of the inequality by 2, a positive number, we have:
$$\begin{aligned}\frac{2}{2}&<\frac{6}{2}\\ 1&<3 \quad \text{which is true}\end{aligned}$$

However, **if we multiply both sides of the inequality by -5, a negative number**, we have:

$$-5(2) < -5(6)$$

$-10 < -30$ which is false!

> **Multiplication/Division by a <u>Negative</u> Number Property**
>
> **If both sides of an inequality are multiplied or divided by the same <u>negative number</u>, the direction of the inequality sign must be <u>reversed</u> (change direction).**
>
> **If $c < 0$ and $a < b$, then $ac > bc$.**

Applying this property to the previous example where we multiplied each side by – 5 we have:

$$2 < 6$$

$$-5(2) < -5(6)$$

$-10 > -30$ if we reverse the direction of the inequality sign, now the result is true!

Example 6:

Solve each of the following inequalities, and graph the solution set on a line graph:

a) $x+3>-5$ **b)** $5x \le -15$ **c)** $5x-3 \ge -23$ **d)** $3-2x \ge -23$

<u>Answers:</u>

a) $x+3>-5$ subtract 3 from each side

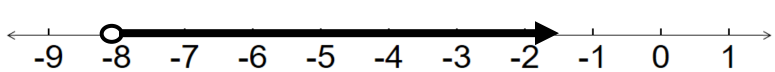
$x>-8$

To **check**, replace x with a number that is greater than -8. An easy number to use, which is greater than -8, is zero.

$0+3>-5$

$3>-5$

The result is a true statement, and therefore the answer is: $x>-8$



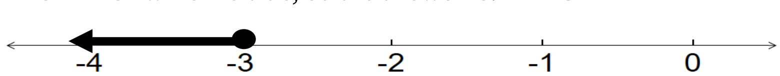
b) $5x \le -15$ divide both sides by 5

$x \le -3$

To **check**, replace x with a number that is less than or equal to -3. This time we cannot use zero, since 0 is not less than or equal to -3. We could use -5:

$5(-5) \le -15$

$-25 \le -15$ which is true, so the answer is: $x \le -3$



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c) To isolate *x*, we add 3 to both sides and then divide both sides by 5.

$5x - 3 \geq -23$

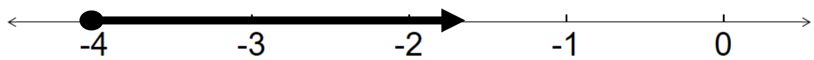
$5x \geq -20$

$x \geq -4$

To **check**, replace *x* with a number that is greater than or equal to – 4. An easy number to use, which is greater than or equal to – 4, is zero.

$5(0) - 3 \geq -23$

$-3 \geq -23$ which is true, so the answer is: $x \geq -4$



d) To isolate *x*, we subtract 3 from both sides and then divide both sides by -2.

$3 - 2x \geq -23$

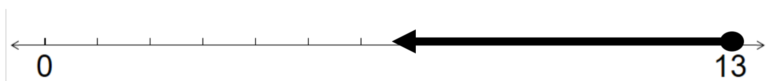
$-2x \geq -26$

$x \leq 13$ the direction of inequality sign is reversed

To **check**, replace *x* with a number that is less than or equal to 13. An easy number to use, which is less than or equal to 13, is zero.

$3 - 2(0) \geq -23$

$3 \geq -23$ which is true, so the answer is: $x \leq 13$



Helpful Tip: If we have a **negative coefficient**, we will need to remember to reverse the direction of the inequality sign when we divide by that negative number in the last step.

Equivalent Inequalities

We can write $2 < 6$ or $6 > 2$. Both statements are true. When using variables, we can write $x < 6$ or $6 > x$. Both statements represent the numbers that are smaller than 6.

To avoid remembering to reverse the direction of the inequality symbol when dividing by a negative coefficient, we can add the negative variable term to both sides of the inequality. By doing that we will make the variable term positive.

Let's try this idea in Example 6d).

$3-2x \geq -23$ add $2x$ to both sides

$3 \geq -23+2x$ add 23 to both sides to isolate the variable term

$26 \geq 2x$

$13 \geq x$

Previously, our answer was $x \leq 13$. Keep in mind that $x \leq 13$ is the same as $13 \geq x$.

Practice Exercises: Solving Inequalities

For exercises 8 – 15, solve each inequality and **graph the solution set on a line graph**. Be sure to **check your answers**.

8) $2x-1>-15$

9) $5-4x \leq -15$

10) $-3x \leq 20$

11) $\frac{3x}{5} \geq -15$

12) $8-x<1$

13) $2(x-1)>-20$

14) $-1>-15+2x$

15) $9x+18 \leq 4(2x-3)$

Example 7:

A student has taken four history tests this semester, and her grades are 80, 92, 74 and 76. She needs to take one more test before she gets her final average. What is the lowest grade she can get on her last test, if she wants her final average to be a B (which is 80) or higher?

Answer:

We know that to get a final average of 80, we will need to add five test grades and divide by 5. We do not know her grade on the last test, so we can call that grade x.

Since her final average will be based on five grades, we can use the inequality:

$\frac{80+92+74+76+x}{5} \geq 80$ To solve, multiply both sides of the inequality by 5

$80+92+74+76+x \geq 400$

$322+x \geq 400$

$x \geq 78$ Therefore the student needs a grade of 78 or higher on the last test to get a final average of B.

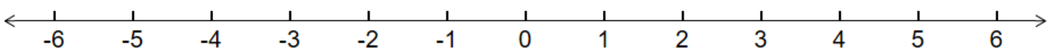
2.5 SUMMARY

1. An ________________ is a statement that one quantity is greater than another. The four inequality symbols are: ____, _____, _____, _____.

<u>Examples</u>:

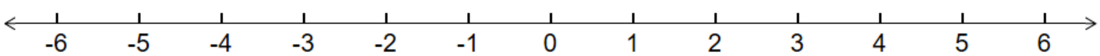
- In symbols, the set of numbers greater than or equal to -2 is written: _________

  On a number line:



- In symbols, the set of numbers less than 3 is written: _________

  On a number line:

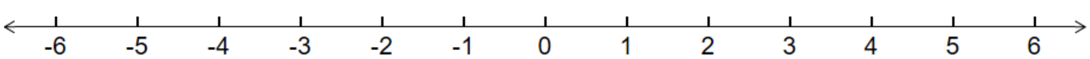


2. A ________________ inequality consists of two inequalities combined.

<u>Example</u>:

- In symbols, the set of numbers between -2 and 3, including -2, is written: ___________

  On a number line:



3. To solve a linear inequality _______________ the inequality sign when dividing or multiplying by a negative number.

<u>Examples</u>: Circle the correct solution:

a) $-2x > 10$

i) $x > -5$ **ii)** $x < 5$ **iii)** $x < -5$ **iv)** $x \le -5$

b) $3 - 3x \le -12$

i) $x \ge -5$ **ii)** $x \le 5$ **iii)** $x \ge 5$ **iv)** $x \le -5$

2.6 Literal Equations

Some equations involve more than one variable. For example, think of formulas you may have learned, such as the area of a rectangle $(A = lw)$ or the formula to change degrees in Fahrenheit to degrees in Celsius $\left(C = \frac{5}{9}(F - 32)\right)$. Such equations are called literal equations.

Variables that are written "In Terms Of" other Variables

If we consider the formula to change degrees Fahrenheit to degrees Celsius $\left(C = \frac{5}{9}(F - 32)\right)$, notice that *C* stands alone on one side. We say that in this formula, degrees Celsius is written "in terms of" degrees Fahrenheit.

You would use this formula if you know that the temperature is $55°F$ and you are asked for that temperature in degrees Celsius. To find degrees Celsius, you would replace *F* with 55.

$$C = \frac{5}{9}(55 - 32)$$
$$C = \frac{5}{9}(23)$$
$$C = 12.8$$
$$\text{Therefore } 55°F \approx 12.8°C$$

But what if you are a tourist in another country where the temperature is given in degrees Celsius? For example, if you are in England and the temperature is $20°C$, do you need a winter coat? (See Example 4 for the answer.) It would be helpful to have a formula for Fahrenheit in terms of Celsius so that you truly understand how warm or cold it is. When we take a formula and rewrite it in terms of another variable in the formula, then we are **solving the formula for a particular variable**.

Solving a Formula

An equation is solved for a particular variable if that variable is isolated on one side, that is, the variable alone equals an expression that does not contain that particular variable.

To solve a formula for a particular variable:
1) **Circle the variable that you are solving for.**
2) **To isolate that variable, use steps for solving an equation to get that particular variable alone on one side.**

Example 1:

For each of the following formulas, determine which variable is solved for and then use the phrase "in terms of" to describe each literal equation.

a) $y = 2x + 5$

b) $d = rt$

c) $I = prt$

d) $y + 1 = x + 2$

Answers:

Formula	**Variable solved for**	**Variable written in terms of**
$y = 2x + 5$	Since y is alone on one side, the formula is solved for y.	y is written in terms of x
$d = rt$	Since d is alone on one side, the formula is solved for d.	d is written in terms of r and t
$I = prt$	Since I is alone on one side, the formula is solved for I.	I is written in terms of p, r, and t.
$y + 1 = x + 2$	Since no variable is alone on one side, this equation is not solved for any variable.	

Example 2:

Solve each formula for the indicated variable.

a) $A = lw$ for w.

b) $D = rt$ for t.

c) $I = prt$ for p.

d) $P = R - C$ for R.

Answers:

a) To isolate the w and write it in terms of A and l, first we circle w. Next, we see that l is connected to w by multiplication. So we divide both sides of the equation by l, and the result is $w = \frac{A}{l}$. The solution follows.

$$\frac{A}{l} = \frac{\cancel{l}w}{\cancel{l}}, \quad \frac{A}{l} = w$$

Helpful Tip: Notice that when solving a formula, the answer is another formula. There is no way to check your answer when solving a formula, so be careful to follow the steps for solving an equation.

b) To isolate the t and write it in terms of D and r, first we circle t. Next, we see that r is connected to t by multiplication. So we divide both sides of the equation by r, and the result is $t = \frac{D}{r}$. The solution follows.

$$\frac{D}{r} = \frac{\cancel{r}t}{\cancel{r}}$$

$$\frac{D}{r} = t$$

c) To isolate the p and write it in terms of I, r, and t, first we circle p. Next, we see that r and t are connected to p by multiplication. So we divide both sides of the equation by r and by t. The result is $p = \frac{I}{rt}$. The solution follows.

$$\frac{I}{rt} = \frac{p\cancel{rt}}{\cancel{rt}}$$

$$\frac{I}{rt} = p$$

d) To isolate the R and write it in terms of P and C, first we circle the R. To get R alone on one side, we add C to both sides of the equation. The result is $R = P + C$.

Example 3:

Solve each formula for the indicated variable.

a) $y = 2x + 5$ for x.

b) $2x - 3y = 6$ for x

c) $2x - 3y = 6$ for y.

Answers:

a) To solve $y = 2x + 5$ for x, circle the x. To isolate the x, first subtract 5 from both sides, then divide both sides by 2.

$$\begin{array}{rl} y & = 2x + 5 \\ -5 & \quad\;\; -5 \\ \hline y - 5 & = 2x \end{array}$$

$$\frac{y-5}{2} = \frac{\cancel{2}x}{\cancel{2}}$$

$$\frac{y-5}{2} = x \quad \text{or} \quad x = \frac{y-5}{2}$$

b) To solve $2x-3y=6$ for x, circle the x. To isolate the x, first add $3y$ to both sides, then divide both sides by 2.

$$2x-3y=6$$
$$2x=6+3y$$
$$\frac{2x}{2}=\frac{6+3y}{2}$$

We can simplify the result, $x=\frac{6+3y}{2}$ by applying the distributive property and dividing every term by 2. In that way, we are making two fractions:

$$x=\frac{6+3y}{2}$$
$$x=\frac{6}{2}+\frac{3y}{2}=3+\frac{3y}{2}$$

Therefore the result can also be written as: $x=3+\frac{3y}{2}$.

c) To solve $2x-3y=6$ for y, circle the y. To isolate the y, first subtract $2x$ from both sides, then divide every term on both sides by -3.

$$2x-3y=6$$
$$\underline{-2x\qquad\quad -2x}$$
$$-3y=-2x+6$$
$$y=\frac{-2x}{-3}+\frac{6}{-3}$$
$$y=\frac{2x}{3}-2$$

A second way to solve $2x-3y=6$ for y would be to add $3y$ to both sides. This makes the coefficient of y positive. This solution follows.

$$2x-3y=6$$
$$\underline{\quad +3y\qquad +3y}$$
$$2x=3y+6$$
$$\underline{-6\qquad -6}$$
$$2x-6=3y$$

Divide every term on both sides by 3, and the result is $\frac{2x-6}{3}=y$ or $y=\frac{2x}{3}-2$.

Some formulas have parentheses, as you will see in Example 4.

Example 4:

Solve the formula, $C = \frac{5}{9}(F - 32)$ for F.

Answer:

To solve $C = \frac{5}{9}(F - 32)$ for F, circle the F. To isolate the F, divide both sides by the factor $\frac{5}{9}$, as in the following solution:

$C \div \frac{5}{9} = \left[\cancel{\frac{5}{9}}(F - 32)\right] \div \cancel{\frac{5}{9}}$ remember that dividing by $\frac{5}{9}$ is the same as multiplying by $\frac{9}{5}$

$\frac{9}{5}C = F - 32$ add 32 to both sides

$F = \frac{9}{5}C + 32$

Now that we have a formula for F in terms of C, we can answer the question from the beginning of this section: If you are in England and the temperature is $20°C$, do you need a winter coat?

$F = \frac{9}{5}C + 32$ replace C with 20

$F = \frac{9}{5}(20) + 32$

$F = 36 + 32$

$F = 68$

Since $20°C = 68°F$ you will not need a winter coat!

Subscripted Variables

We know that variables are often alphabet letters that represent constants. But some formulas may have more than one constant that must be used for the same variable, such as x. To help distinguish between variables that may both be called x, we use subscripted variables. For the variable x, we have x_0 read as "x sub 0". Note that x and x_0 are two different variables that represent two different constants, similar to x and y. Other subscripted variables with x are $x_1, x_2, x_3, x_4, \ldots$ read as x sub 1, x sub 2, x sub 3, x sub 4, and so on. Interestingly, there are an infinite number of subscripted variables using x. In the same way, we have an infinite number of subscripted variables for y, and for every letter in the alphabet.

Example 5:

Solve the equation $y - y_1 = m(x - x_1)$ for m.

Answer:

To solve $y - y_1 = m(x - x_1)$ for m, circle the m. To isolate the m, divide both sides by the factor $(x - x_1)$, as in the following solution:

$$(y - y_1) = m(x - x_1)$$

$$\frac{(y - y_1)}{(x - x_1)} = m$$

Practice Exercises: Solving Literal Equations

For exercises 1 – 8, solve for the indicated variable.

1) $A = \frac{1}{2}bh$ for b

2) $P = 2l + 2w$ for l

3) $2x - 3y = 6$ for y

4) $3x - 2y = 6$ for y

5) $C = 2\pi r$ for r

6) $C_1 V_1 = C_2 V_2$ for C_1

7) $A = P(1 + rt)$ for r

8) $-4x - 3y = 9$ for y

2.6 SUMMARY

1. Use the word bank to fill in the blanks below.

- ________________ are used frequently in everyday life, including geometry calculations, sports, and interest.
- A formula is an ________________________ .
- Solving for a variable in an equation means it is isolated and written ______________________ the other variables.
- The ____________________ property can be applied to rewrite fractions with more than one term in the numerator.
- A subscripted variable represents a _______________ value, often an initial value of the variable.

Word Bank
distributive
equation
formulas
constant
in terms of

2. State what operation (add, subtract, multiply or divide) is needed to solve for y:

a) To solve $V = xyz$ for y, ______________ both sides by ___ and ____.

b) To solve $2x - 3y = 6$ for y, ___________ $2x$ on both sides and ________ by -3.

2.7 Linear Functions: Intercepts

Solutions and Lines

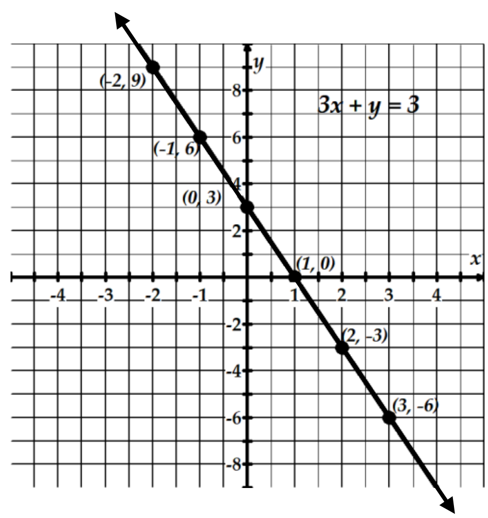
In this section, we continue to discuss solving equations. We now focus on linear equations in two variables. The solutions to linear equations in two variables are written as ordered pairs. Hence, the solutions can be represented by points in the plane. We also note that the phrase "graph the equation" means to plot the solutions (points).

Consider the equation $3x + y = 3$. We will graph six solutions (ordered pairs) to this equation on the coordinate system below. We will find the solutions by choosing x-values (from –2 to +3), substituting them into the equation $3x + y = 3$ and then solving to obtain the corresponding y-coordinates. We can keep track of the ordered pairs by using a table.

Look back at tables we created in Sections 2.1 and 2.2. Each rule was written as "y = " or "$f(x)$ =." As we build a table, it may be easier to find the y-coordinate of each ordered pair if we solve the equation for y as we did in the previous section. To solve $3x + y = 3$ for y, subtract $3x$ from both sides. The equivalent equation is: $y = -3x + 3$.

The numerical and graphical representations of the solutions to $3x + y = 3$ follow.

x	y: $y = -3x + 3$	(x, y)
–2	$-3(-2)+3=9$	$(-2, 9)$
–1	$-3(-1)+3=6$	$(-1, 6)$
0	$-3(0)+3=3$	$(0, 3)$
1	$-3(1)+3=0$	$(1, 0)$
2	$-3(2)+3=-3$	$(2, -3)$
3	$-3(3)+3=-6$	$(3, -6)$



We have plotted only six solutions to the equation $3x + y = 3$. We could have included many more values of x. In fact, there are infinitely many solutions. By observing the six points we have plotted, we can speculate as to the location of all the other points. The six points we plotted seem to lie on a straight line. This would lead us to believe that all the other points (solutions) also lie on that same line. To show that there are infinitely many solutions, we connect the points and extend the line beyond our points. We can place an arrow at each end of the line.

General Form of a Linear Equation in Two Variables

We say the equation $3x + y = 3$ is the general form of this linear equation.

General Form of a Linear Equation in Two Variables

There is a standard form in which linear equations in two variables are written. Suppose that *a, b,* and *c* are any real numbers and that *a* and *b* cannot both be zero at the same time. Then, the linear equation in two variables $ax + by = c$ is said to be in General Form.

We must stipulate that *a* and *b* cannot both equal zero at the same time, for if they were we would have $0x + 0y = c$; or $0 = c$ which is only true if $c = 0$ and does not give us points for a line.

To use a table with six points to graph this line, we rewrote the general form and solved for *y*. But perhaps there is an easier way to graph a line in standard form.

Graphing a Line Using the Horizontal and Vertical Intercepts

What is the minimum number of points needed to graph a straight line? If you have two points, you can plot them, connect them using a straightedge or ruler, and extend the line.

Look back at the graph of $3x + y = 3$. Notice where the line crosses the *x*-axis (horizontal axis) and the *y*-axis (vertical axis). These two special points are called the horizontal axis and vertical axis intercepts, or the *x*- and *y*-intercepts.

***x*-intercept (horizontal axis intercept)**

The point at which the line crosses the *x*-axis is called the *x*-intercept and the *y*-value at that point is zero.

***y*-intercept (vertical axis intercept)**

The point at which the line crosses the *y*-axis is called the *y*-intercept. The *x*-value at this point is zero (since the point is neither to the left nor right of the origin).

How to find the *x*- and *y*-intercepts:

1) **The *x*-intercept can be found by substituting the value 0 for *y* into the equation and solving for *x*.**
2) **The *y*-intercept can be found by substituting the value 0 for *x* into the equation and solving for *y*.**

Let's graph the line $3x + y = 3$ by finding and plotting only two points: the x- and y-intercepts.

To find the x-intercept, let $y = 0$:

$$3x + (0) = 3$$
$$3x = 3$$
$$x = 1$$

The x-intercept is (1, 0). Look back at the graph. Did the line $3x + y = 3$ cross the x-axis at $x = 1$?

To find the y-intercept, let $x = 0$:

$$3(0) + y = 3$$
$$y = 3$$

The y-intercept is (0, 3). Look back at the graph. Did the line $3x + y = 3$ cross the y-axis at $y = 3$?

Helpful Tip: The x- and y-intercepts are **points**. It is helpful to write the format for the answers, even before you find the points.

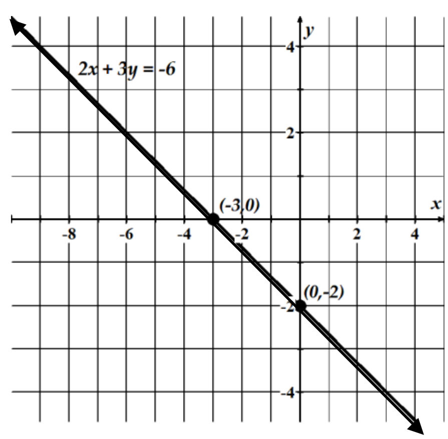
The format for the x-intercept is $(a, 0)$ such that a is where the line crosses the x-axis.
The format for the y-intercept is $(0, b)$, such that b is where the line crosses the y-axis.

Example 1:

Find the x- and y-intercepts for the line: $2x + 3y = -6$. Graph the line by plotting the two points.

Answer:

To find the x-intercept, let $y = 0$: $2x + 3(0) = -6$ $2x = -6$ $x = -3$ The x-intercept is $(-3, 0)$.	To find the y-intercept, let $x = 0$: $2(0) + 3y = -6$ $3y = -6$ $y = -2$ The y-intercept is $(0, -2)$.



Practice Exercises: Graphing a line using the intercept method

For exercises 1 – 4, find the x- and y-intercepts. Write both intercepts as ordered pairs. Then graph the line on graph paper, by plotting the two points.

1) $3x + y = -6$

2) $2x - 5y = 10$

3) $2x - 3y = -3$

4) $4x - 6y = -12$

Interpreting Horizontal and Vertical Axis Intercepts in Applications

Example 2:

A paper airplane is launched from a height of 3 feet (36 inches) and descends at a rate of 4 inches per second. If y is the height of the paper airplane in inches and x is the time in flight in seconds, then the formula that represents the height of the paper airplane (in inches) is $y = -4x + 36$.

a) Find the horizontal axis intercept and interpret its meaning in the context of this problem.

b) Find the vertical axis intercept and interpret its meaning in the context of this problem.

<u>**Answers**</u>:

a) To find the horizontal axis intercept, let $y = 0$:

$$\begin{aligned} y &= -4x + 36 \\ 0 &= -4x + 36 \quad \text{add } 4x \text{ to both sides} \\ 4x &= 36 \\ x &= 9 \end{aligned}$$

The horizontal axis intercept is (9, 0). Since the x values are flight time in seconds, the meaning of this intercept is that at 9 seconds, the height of the plane is 0 inches, so it has landed.

b) To find the vertical axis intercept, let $x = 0$:

$$\begin{aligned} y &= -4(0) + 36 \\ y &= 36 \end{aligned}$$

The vertical axis intercept is (0, 36). Since the y values are in inches, the meaning of this intercept is that at 0 seconds, the height is 36 inches.

Practice Exercise: Interpreting horizontal and vertical axis intercepts in an application

For exercise 5, answer the following questions.

5) Julio has received a $40 gift card to the local movie theater. He plans to see a movie every week on Tuesdays, when the price to see a movie is $5 all day.

a) If y represents the amount of money remaining on the gift card, and x represents how many movies he has seen, write the rule that models this situation.

b) Find the horizontal axis intercept and interpret its meaning in the context of this problem.

c) Find the vertical axis intercept and interpret its meaning in the context of this problem.

Using Function Notation to Find Horizontal and Vertical Axis Intercepts

In Section 2.1 we discussed how to find $f(x)$ if given x, and how to find x if given $f(x)$. We can now continue this discussion to determine how function notation can be used to find horizontal and vertical axis intercepts.

How to find the x- and y-intercepts when using function notation:

1) Since the x-intercept can be found by substituting the value 0 for y into the equation and solving for x, in function notation this is written as: find x if $f(x)=0$.

2) Since the y-intercept can be found by substituting the value 0 for x, in function notation the y-intercept is $f(0)$.

Helpful Tips: In function notation,

1) When finding the x-intercept, replace $f(x)$ with 0 and **solve the equation** for x.

2) When finding the y-intercept, **use substitution** to find $f(0)$.

Exmple 3:

For each of the following functions, find the x- and y-intercepts. Write both intercepts as ordered pairs. Then graph the line on graph paper, by plotting the two points.

a) $f(x)=3x-9$ **b)** $f(x)=8-2x$

Answers:

a) $f(x) = 3x - 9$

To find the *x*-intercept, let $f(x) = 0$:

$0 = 3x - 9$ add 9 to both sides

$9 = 3x$

$3 = x$

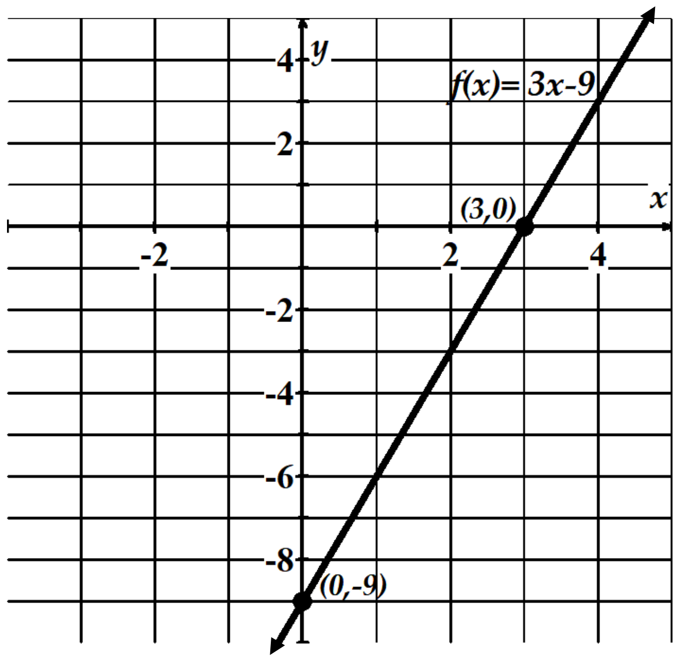
The *x*-intercept is (3, 0).

To find the *y*-intercept, let $x = 0$:

$f(0) = 3(0) - 9$

$f(0) = -9$

The *y*-intercept is (0, – 9).



b) $f(x) = 8 - 2x$

To find the *x*-intercept, let $f(x) = 0$:

$0 = 8 - 2x$ add $2x$ to both sides

$2x = 8$

$x = 4$

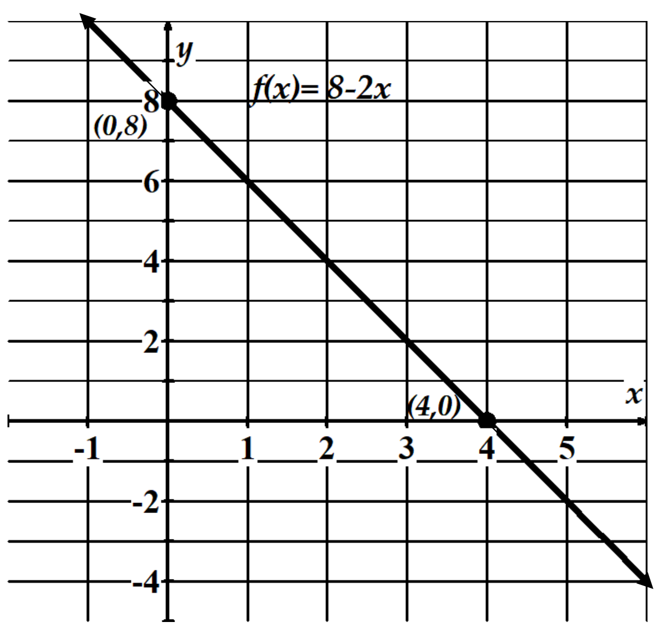
The *x*-intercept is (4, 0).

To find the *y*-intercept, let $x = 0$:

$f(0) = 8 - 2(0)$

$f(0) = 8$

The *y*-intercept is (0, 8).



Practice Exercises: Using function notation to find x- and y-intercepts

For exercises 6 – 10, find the x- and y-intercepts. Write both intercepts as ordered pairs. Then **graph the line on graph paper, by plotting the two points.**

6) $f(x) = 2x + 4$

7) $f(x) = 12 - 3x$

8) $f(x) = 5 - x$

9) $f(x) = 2x - 7$

10) $f(x) = 5x$

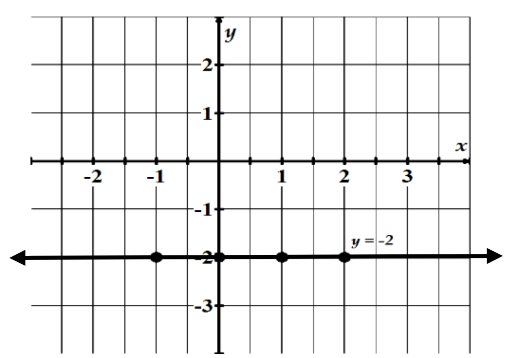
Equations of Horizontal and Vertical Lines

In all the graphs we have observed so far, the lines have been slanted. This will always be the case when **both** variables (x and y) appear in the equation. If only one variable appears in the equation, then the line will be either vertical or horizontal. To see why, let's consider a specific case.

Using the general form of a line, $ax + by = c$, we can produce an equation with exactly one variable by choosing $a = 0$, $b = -4$, and $c = 8$. The equation $ax + by = c$ becomes $0x - 4y = 8$. Because $0x$ is 0, this equation can be simplified to $-4y = 8$. Dividing both sides by – 4, gives the equivalent equation $y = -2$.

This means that regardless of which number we choose for x, the corresponding y-value is -2. Let's look at a table of points on this line and the graph of the plotted points.

x	y: $y = -2$	(x, y)
– 1	– 2	(– 1, – 2)
0	– 2	(0, – 2)
1	– 2	(1, – 2)
2	– 2	(2, – 2)



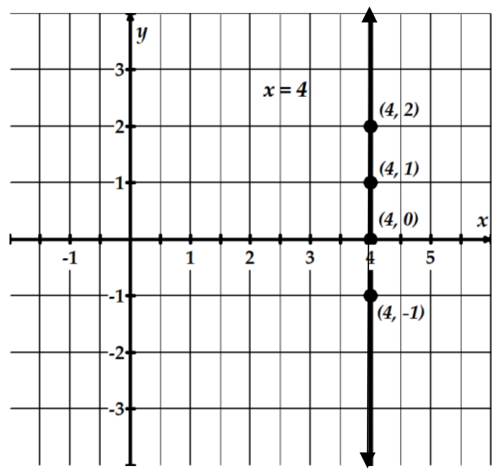
The Equation of a Horizontal Line

The equation of a horizontal line has the form: $y = b$.
The horizontal line will pass through b on the y-axis; (0, b) is the y-intercept.

The equation of a vertical line can be derived in a similar way. Using the general form of a line, $ax + by = c$, we can produce an equation with exactly one variable by choosing $a = 2,\ b = 0,$ and $c = 8$. The equation $ax + by = c$ becomes $2x + 0y = 8$. Because $0y$ is 0, this equation can be simplified to $2x = 8$. Dividing both sides by 2 gives the equivalent equation $x = 4$. This means that x can only be 4.

Let's look at a table of points on this line, where x can only be replaced with 4 and y can be any real number. The graph of the plotted points is on the right.

x $x = 4$	y	(x, y)
4	–1	(4, –1)
4	0	(4, 0)
4	1	(4, 1)
4	2	(4, 2)



The Equation of a Vertical Line

The equation of a vertical line has the form: $x = a$.
The vertical line will pass through a on the x-axis; (a, 0) is the x-intercept.

Helpful Tips:

A horizontal line passes through b on the y-axis and has no x-intercept.

A vertical line passes through a on the x-axis and has no y-intercept.

Example 4:

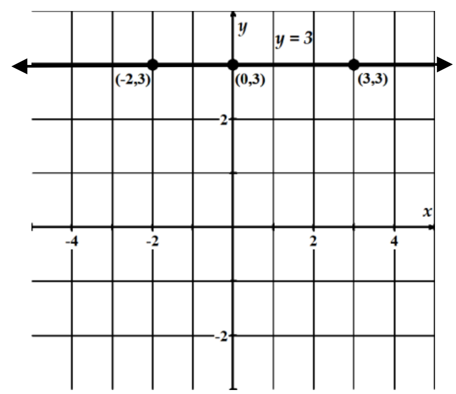
Write an equation of the line that meets the following conditions, and graph each line.

a) Horizontal line passing through (– 2, 3).
b) Vertical line passing through (– 2, 3).
c) Horizontal line passing through (0, – 1).
d) Vertical line passing through (– 1, 0).

<u>**Answers**</u>**:**

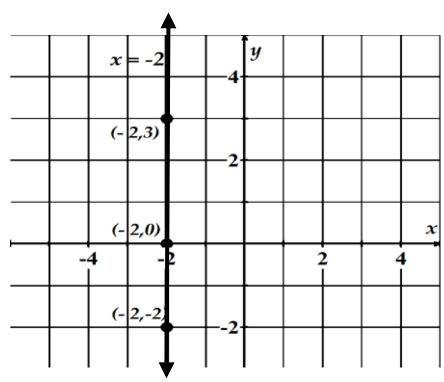
a)

The horizontal line crosses the y-axis at $y = 3$, which is the equation of this line.



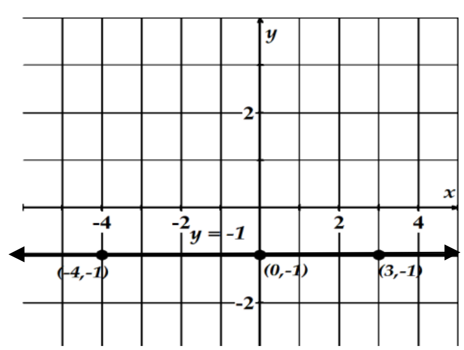
b)

The vertical line crosses the x-axis at $x = -2$, which is the equation of this line.



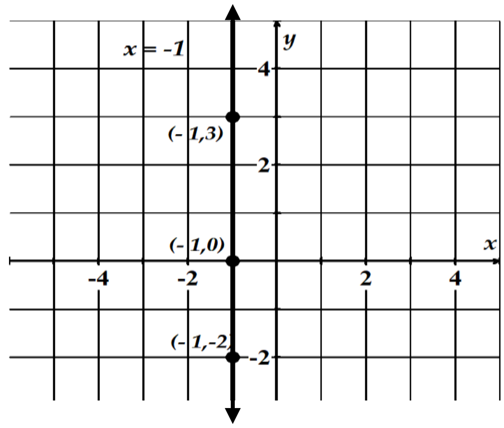
c)

The horizontal line crosses the y-axis at $y = -1$, which is the equation of this line.



d)

The vertical line crosses the x-axis at $x = -1$, which is the equation of this line.



Practice Exercises: Writing an equation of a horizontal and a vertical line

For exercises 11 – 20, write an equation of the line that meets the following conditions and sketch the graph of each line.

11) Horizontal line passing through $(-2, -2)$.

12) Vertical line passing through $(-3, -2)$.

13) Horizontal line passing through $(-8, 0.5)$.

14) Horizontal line passing through $(-3, -2)$.

15) Horizontal line passing through $(-2, 0)$.

16) Vertical line passing through $(-2, 0)$.

17) Vertical line passing through $(0, -2)$.

18) Vertical line passing through $(-2, -2)$.

19) Vertical line passing through $(-8, 0.5)$.

20) Horizontal line passing through $(-4, 4)$.

2.7 SUMMARY

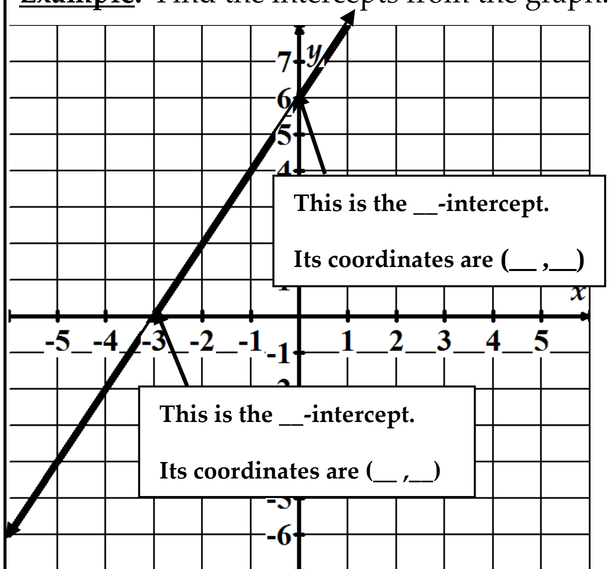
1. The **y-intercept (vertical intercept)** of a function is the point where the graph crosses the ___- axis.

To find the y-intercept from an equation, substitute 0 for ___ and solve for ___.

2. The **x-intercept (horizontal intercept)** of a function is the point where the graph crosses the ___- axis.

To find the x-intercept from an equation, substitute 0 for ___ and solve for ___.

Example: Find the intercepts from the graph.



Example: Find the intercepts of the equation $3x - y = 9$. Write each as an ordered pair.

y-intercept :

Set $x = 0$.

$3(0) - y = 9$

x-intercept :

Set $y = 0$.

$3x - 0 = 9$

Write the x and y-intercepts of the equation $3x - y = 9$ in the following input-output table:

x	y	(x, y)
0		
	0	

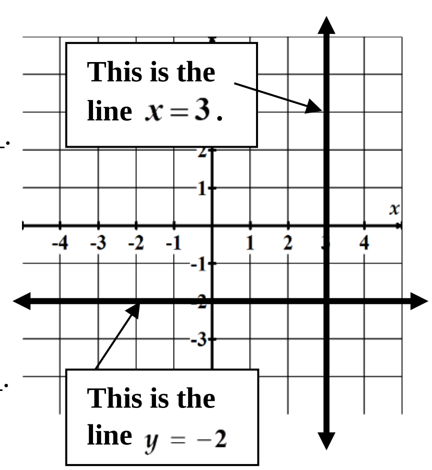
3. For the function $f(x) = -4x + 12$,

a) Finding $f(0) =$ ________ gives the ___-intercept. This is the point (____,____).

b) Solving $f(x) = 0$ gives the ___-intercept. This is the point (____,____).

4. Horizontal Lines

- On a horizontal line, all of the ___ - coordinates are the same.
- The equation of a horizontal line through the point (a, b) is _____.
- The equation of the horizontal line that passes through $(3, -2)$ is______.



5. Vertical Lines

- On a vertical line, all of the ___ - coordinates are the same.
- The equation of a vertical line through the point (a, b) is _______.
- The equation of the vertical line that passes through $(3, -2)$ is______.

2.8 Linear Functions: Slope and Rate of Change

As we graph lines, we will want to be able to identify different properties of the lines we graph. One of the most important properties of a line is its slope.

Introduction to Slope

❖ **Slope is a Measure of Steepness**

Have you ever gone skiing, biking, or skateboarding? If so, steepness of the terrain is an important component of each sport. A steep mountain would have a large slope, such as 25. Cross-country skiing on an almost level trail would have a small upward slope, such as $\frac{1}{10}$.

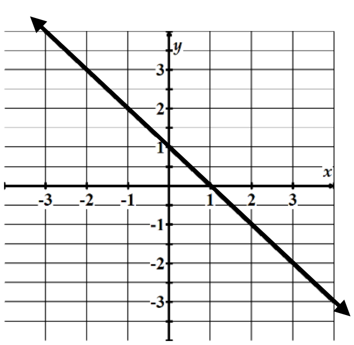
As we measure the steepness of a line, we are interested in how fast the line rises compared to how far the line runs.

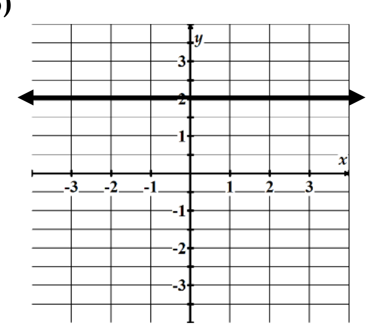
❖ **Slope is a Measure of Direction**

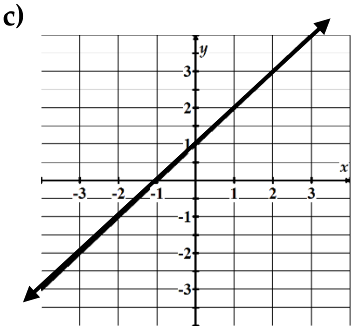
A line that goes up **from left to right** (rising) has a positive slope, and a line that goes down **from left to right** (falling) has a negative slope. A horizontal line is neither going uphill (positive slope) or downhill (negative slope). So a horizontal line has a slope that is zero.

Example 1:

For each of the following graphs, determine whether the slope is positive, negative, or zero.

a)


b)


c)


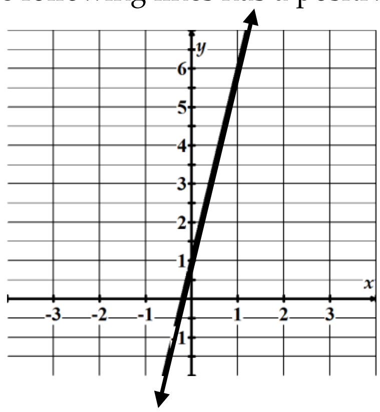
__Answers__:

a) The slope of this line is negative, since from left to right the line is falling.
b) The slope of this line is zero, since the line is horizontal.
c) The slope of this line is positive, since from left to right the line is rising.

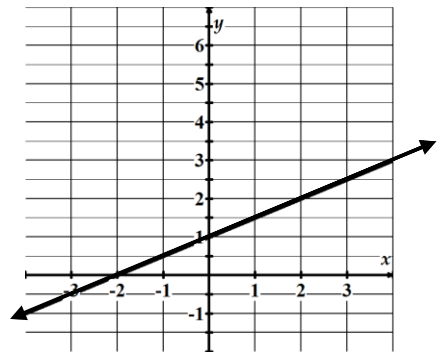
Example 2:

Each of the following lines has a positive slope. Circle the graph with the steeper slope.

a)



b)



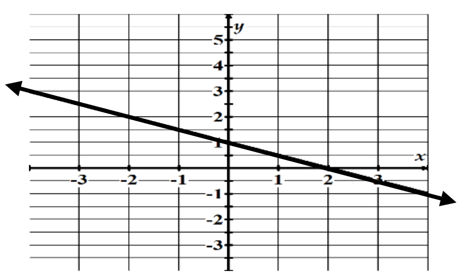
Answer:

The graph in part a) has a steeper positive slope.

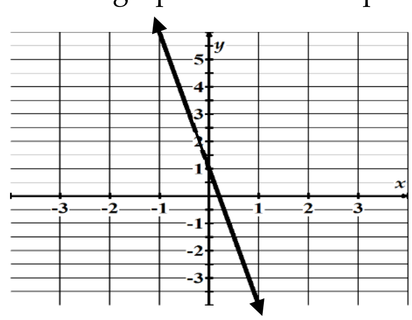
Example 3:

Each of the following lines has a negative slope. Circle the graph with the steeper slope.

a)



b)



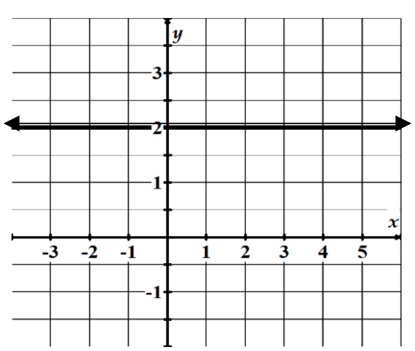
Answer:

The graph in part b) has a steeper negative slope.

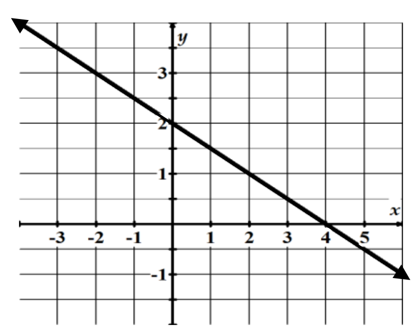
Practice Exercises: Slope direction and steepness

For exercise 1, determine whether the slope is positive, negative, or zero.

1) a)

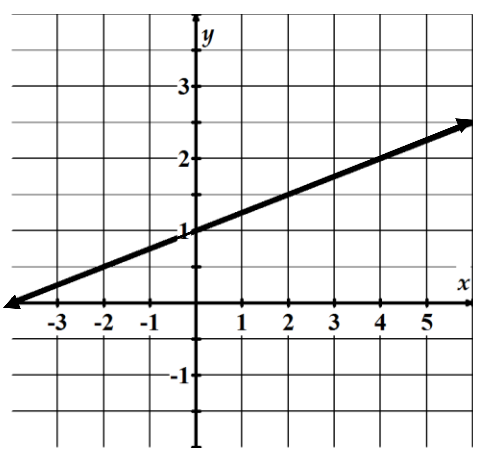


b)

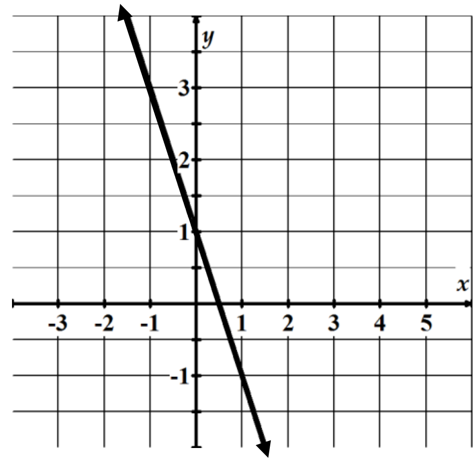


For exercise 2, determine which graph is steeper. *Hint:* notice that the scales are the same. Think about which line is steeper, regardless of the direction.

2) a)



b)



How to Find the Slope of a Line from the Graph

If you are given the graph of a line, you can use the following two facts to determine the slope of that line.

1. **Slope can be described by the fraction:** $\dfrac{\text{rise}}{\text{run}}$

As we measure steepness we are interested in how fast the line rises compared to how far the line runs. For this reason we will describe slope as the fraction: $\dfrac{\text{rise}}{\text{run}}$.

2. **Slope can be measured by the fraction:** $\dfrac{\text{change in } y}{\text{change in } x}$

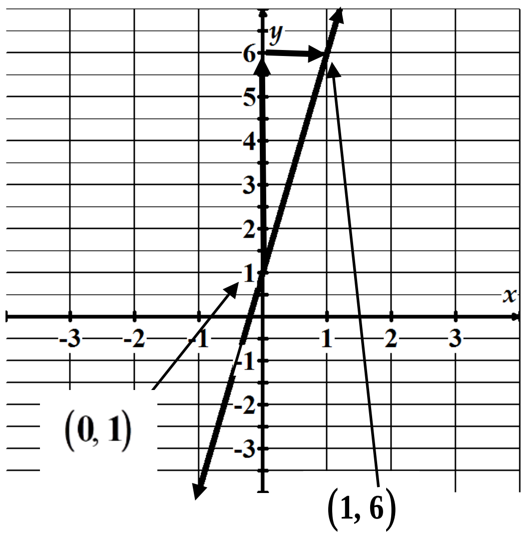
Rise is the vertical change, or a change in the y-values. Run is the horizontal change, or a change in the x-values.

To find the slope of a line from the graph:

1) **Locate two integer points on the graph.**
2) **Begin at the point on the left. Decide how any spaces up or down (rise), and then how many spaces to the right (run) you would travel to get to the point on the right. Be sure to pay attention to the scale on each axis.**
3) **The slope is written as a fraction:** $\dfrac{\textbf{rise}}{\textbf{run}}$.

Example 4:

Find the slope of the following graph.

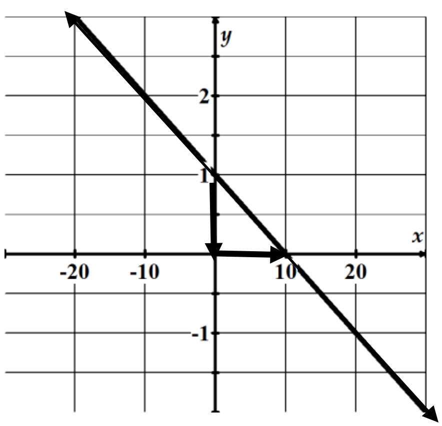


Answer:

The point on the left is $(0, 1)$ and the point on the right is $(1, 6)$. To move from $(0, 1)$ to $(1, 6)$, we would go up 5 and over to the right 1. Therefore the slope is $\frac{5}{1}=5$.

Example 5:

Find the slope of the following graph.



Answer:

The point on the left is $(0, 1)$ and the point on the right is $(10, 0)$. To move from $(0, 1)$ to $(10, 0)$, we would go down 1 and over to the right, 10. Therefore the slope is $\frac{-1}{10}=-\frac{1}{10}$.

Helpful Tip:

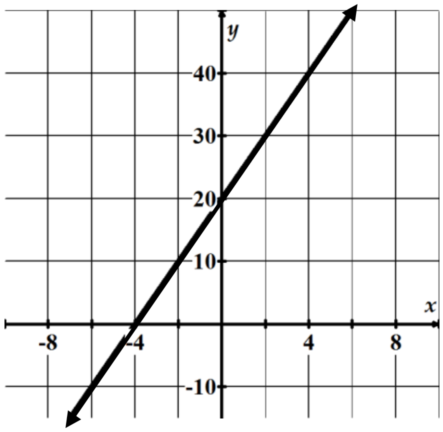
A negative fraction can be written in three ways: $\frac{-1}{10}=\frac{1}{-10}=-\frac{1}{10}$.

When the slope of a line is a negative fraction, we write it with the negative sign in the numerator, which means that to get from the first point to the next point on a line, we will move down and then over to the right.

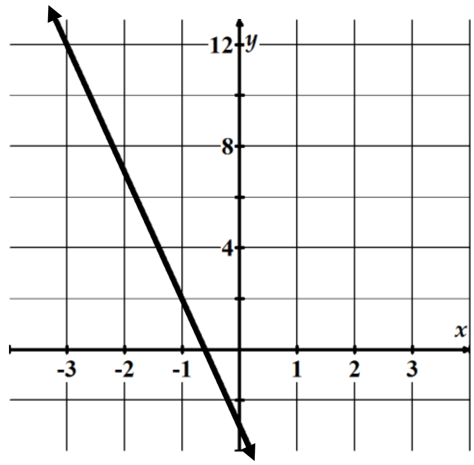
Practice Exercises: Finding the slope of a line from the graph

For exercises 3 – 4, find the slope of each line. Use two integer points. Watch the scale!

3)



4)



Formula for the Slope of a Line

We can find the slope of a line through two points without seeing the points on a graph. We can do this using a slope formula. If the rise is the change in y values, we can calculate this by subtracting the y values of a point. Similarly, if run is a change in the x values, we can calculate this by subtracting the x values of a point. In this way we get the following Slope Formula.

- **Slope Formula**

The slope of a line through two points, (x_1, y_1) and (x_2, y_2), is $\dfrac{y_2 - y_1}{x_2 - x_1}$.

Note: Since the coordinates of both points are (x, y), we use subscripted variables to distinguish between the coordinates of the first point and the coordinates of the second point.

- **Using Δ to represent the difference**

When we find $\dfrac{\text{change in } y}{\text{change in } x}$ we are subtracting the y values in the numerator and the x values in the denominator. The change in y is the difference of the y values, and the change in x is the difference of the x values. We use the Greek letter delta, Δ, to represent the word "difference."

- **Slope is represented by the variable *m***

When mathematicians began working with slope, it was called the modular slope. This may be why we often represent the slope with the variable m.

Summary of Slope Notation and the Slope Formula

$$\textbf{Slope} = m = \frac{\textbf{rise}}{\textbf{run}} = \frac{\textbf{change in } y}{\textbf{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Helpful Tips:

1) Be sure to write the subtraction in the formula. If a coordinate is negative, then there will be two negative signs.
2) When you begin to subtract, if you write the y_1 coordinate first, then the denominator must have the x_1 coordinate written first.
3) Remember that for slope, the **numerator must have the change in *y***, which is $y_2 - y_1$.

Example 6:

Find the slope of each line that passes through the given points. Then state whether the line is rising or falling.

a) (– 10, 8) and (2, 2)
b) (– 10, – 3) and (– 4, 3)

Answers:

a) It may be helpful to identify $x_1, y_1, x_2,$ and y_2:

x_1	y_1
–10	8

x_2	y_2
2	2

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-8}{2-(-10)} = \frac{-6}{12} = -\frac{1}{2}$$

The slope is negative, so the line is falling.

b) Identify $x_1, y_1, x_2,$ and y_2:

x_1	y_1
–10	–3

x_2	y_2
– 4	3

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-(-3)}{-4-(-10)} = \frac{6}{6} = \frac{1}{1} = 1$$

The slope is positive, so the line is rising.

Helpful Tip:

You can check your answer by plotting the two points on graph paper and looking at the resulting graph to see if it is rising or falling.

The Slope of a Horizontal Line and a Vertical Line

We have already seen in Example 1 that **the slope of a horizontal line is zero**. Now, let's use the **slope formula** to find the slope of a horizontal line and the slope of a vertical line.

Example 7:

Find the slope of the line connecting the points (– 8, 3) and (2, 3).

Answer:

Identify $x_1, y_1, x_2,$ and y_2:

x_1	y_1
–8	3

x_2	y_2
2	3

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{3-(3)}{2-(-8)}=\frac{0}{10}=0$$

The line is horizontal, and the slope is zero.

Example 8:

Find the slope of the line connecting the points $(-2, 3)$ and $(-2, -3)$.

Answer:

Identify $x_1, y_1, x_2,$ and y_2:

x_1	y_1
–2	3

x_2	y_2
–2	–3

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{-3-(3)}{-2-(-2)}=\frac{-6}{0}$$

The slope is undefined, so there is no slope.
The line is vertical.

There is a big difference between no slope and a zero slope. Zero is an integer and it has a value. No slope has no value, it is undefined

- The slope of a flat horizontal line is zero.
- A vertical line has no slope.

Helpful Tips:

To help you remember slope, think of two points A and B. To get from point A to point B, you can walk:

Uphill (Positive slope) B A	Downhill (Negative slope) A B	Horizontally (Zero slope) A B	But you cannot walk vertically (no slope) B A

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Practice Exercises: Finding the slope of a line using the Slope Formula

For exercises 5 – 10, find the slope of each line that passes through the given points. Then state whether the line is rising, falling, horizontal or vertical.

5) $(-1, -3)$ and $(2, 3)$

6) $(4, -8)$ and $(4, 2)$

7) $(-3, 2)$ and $(-5, 2)$

8) $(3, 1)$ and $(-2, 2)$

9) $(8, -3)$ and $(-2, 2)$

10) $(4, -1)$ and $(2, -1)$

Applications with Slope as Rate of Change

You will often see applications in real life that discuss how a change in one item affects a change in another item. For example, if three books cost \$30, then it can be determined that the rate of change in the price of books is \$10 per book. This means that the price increases by \$10 (change in y) for each book purchased (change in x). We can also say that the rate of change in the price is \$10/book. We write slope as a unit rate, when it is appropriate to have 1 in the denominator.

We have previously seen that slope is the $\frac{\text{change in } y}{\text{change in } x}$.

Slope = Rate of Change = $\boldsymbol{\frac{\text{change in } y}{\text{change in } x}}$

Sometimes, we can collect data to determine the $\frac{\text{change in } y}{\text{change in } x}$ and summarize the data in a table, as in the next example.

Example 9:
A paper airplane is launched from a height of 5 feet (60 inches). The height was taken every second until the airplane landed. The following table was created in which x represents the time in seconds, and y represents the airplane's height in inches.
Use the following data to find the rate of speed in inches per second of the airplane, and interpret it as a rate of change.

x: time in seconds	y: height in inches	(x, y)
0	60	(0, 60)
1	50	(1, 50)
2	40	(2, 40)
3	30	(3, 30)
4	20	(4. 20)
5	10	(5, 10)
6	0	(6, 0)

Answer:

To find the rate of speed in inches per second, select any two points from the table and use the slope formula to find the slope. Let's use the points (5, 10) and (6, 0).

Identify $x_1, y_1, x_2,$ and y_2:

x_1	y_1
5	10

x_2	y_2
6	0

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 10}{6 - 5} = \frac{-10}{1} = -10$$

Since slope $= \frac{\text{change in } y}{\text{change in } x}$, we use the units for y over the units for x. Therefore the airplane's rate of speed is -10 inches per second. In other words, we can interpret the rate of change by stating that the plane is descending at a rate of 10 inches per second.

Helpful Tip:

When finding the slope of a line from a table, you can use any two points on the table. In fact, if you select several different pairs of points and find the slope for each pair, the result will be the same constant value.

Example 10:

Bobby opened a savings account and decided to deposit the same amount each month. After 3 months, he had $225 in his account, and after 5 months he had $375.

a) Find the average rate of change in his savings account per month.

b) Write a rule for this problem, which can be used to find how much money will be in the account after 1 year.

<u>Answer</u>:

a) First we define our variables. To find savings per month, then the savings will be y and the months will be x. (Remember that for slope, we have $\frac{\text{change in } y}{\text{change in } x}$).

Now we can write our two points: (3, 225) and (5, 375).

Identify x_1, y_1, x_2, and y_2:

x_1	y_1
3	225

x_2	y_2
5	375

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{375 - 225}{5 - 3} = \frac{150}{2} = 75$$

The savings increase at a rate of $75/month.

b) If savings increase at a rate of $75/month, then the rule for this table is $y = 75x$.

In 1 year (12 months) Bobby will have 75(12) or $900 in the savings account.

Helpful Tip:

Slope is a unit rate, that is, we write the rate of change with the denominator = 1. So in Example 10, rather than leaving the answer as $150 per 2 months, we simplify and write it as $75/month.

Practice Exercises: Applications with slope as a rate of change

For exercises 11 – 12, find the slope and interpret the slope as a rate of change.

11) A student purchases gasoline on two different days. The first day, he buys 4 gallons for a cost of $9.16. A week later, he buys 7 gallons and pays $16.03. Find the average rate of change in the cost of gasoline per gallon.

12) The following table records the number of M&M's and their calories. Find the slope and interpret the slope as a rate of change. Include units.

x: number of M&M's	*y:* calories	(*x*, *y*)
14	63	(14, 63)
42	189	(42, 189)

2.8 SUMMARY

1. **Slope, *m*,** is a measure of _________________ and ________________ of a line.

2. We will find slope in three ways:

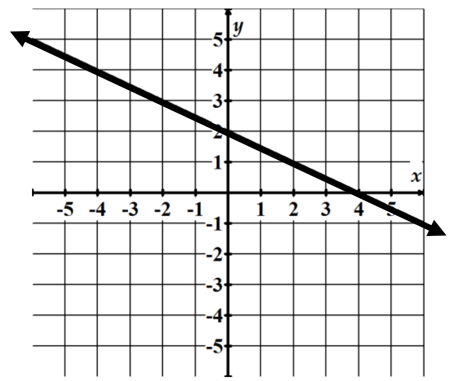
   ➢ **Given a table**, find $\dfrac{\Delta\_\_}{\Delta\_\_}$ for several different pairs of rows of the table. If this value is constant, then it is the slope of the line.

   Example: Find the slope of the line given by the table.

x	y	(x, y)
-2	7	(–2, 7)
0	1	(0, 1)
3	-8	(3, –8)
8	-23	(8, –23)

   ➢ **Given a graph**, using two points with integer coordinates, count the $\dfrac{\Delta\_\_}{\Delta\_\_}$ or $\dfrac{\text{rise}}{\text{run}}$ to get the slope.

   Example: Find the slope of the line graphed to the right.



   ➢ **Given 2 points** (x_1, y_1), and (x_2, y_2) the slope formula is $m = \underline{\qquad\quad} = \dfrac{\Delta\_\_}{\Delta\_\_}$ or $\dfrac{\text{rise}}{\text{run}}$.

   Example: Find the slope of the line passing through (2, 3) and (– 4, 6).

3. The **direction** of a line:
   - **Positive** slope means_____________ (rise or fall) from left to right
   - **Negative** slope means_____________ (rise or fall) from left to right
   - **Zero** slope means ____________________(horizontal or vertical)
   - **No (undefined)** slope means ____________________(horizontal or vertical)

4. **Slope** is also called the **average rate of change** in applications where the input and output variables have units attached to them (miles, dollars, inches, gallons, etc.)
   The units for slope are ____ per ____

   Units for __________ Units for __________

2.9 Linear Functions: Slope-Intercept Form

In Section 2.7, we graphed the equation $3x + y = 3$ by making a table. We realized that to complete the table, it would be easier to find the y values if the equation was solved for y, as in Example 1 below. In this section, we will use the table and graph of lines to find both the y-intercept (vertical axis intercept) and slope of the line.

Finding the y-Intercept and Slope from a Table and Graph

Example 1:

a) Use the table for $3x + y = 3$ to find the y-intercept and slope of the line.

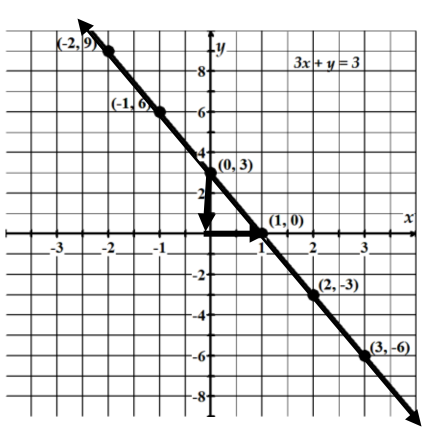
x	y: $y = -3x + 3$	(x, y)
–2	$-3(-2)+3=9$	$(-2, 9)$
–1	$-3(-1)+3=6$	$(-1, 6)$
0	$-3(0)+3=3$	$(0, 3)$
1	$-3(1)+3=0$	$(1, 0)$
2	$-3(2)+3=-3$	$(2, -3)$
3	$-3(3)+3=-6$	$(3, -6)$

<u>**Answer**</u>:

The y-intercept has $x = 0$. The point in the table that has $x = 0$ is (0, 3).
To find the slope, select two points. We can use the x- and y-intercept points: (0, 3) and (1, 0).

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-3}{1-0} = \frac{-3}{1} = -3$$

b) Use the graph for $3x + y = 3$
to find the slope and y-intercept of the line.



<u>**Answer**</u>:

The y-intercept is the point where the line crosses the y-axis: (0, 3).
To find the slope, select two points and find $\frac{\text{rise}}{\text{run}}$. We can use the x- and y-intercept points: (0, 3) and (1, 0).
To go from (0, 3) to (1, 0), we move down 3 spaces and over to the right 1.
Therefore the slope is $\frac{-3}{1} = -3$.

In the next few examples we will find the y- intercept and slope from either the table or the graph, so you should be able to do both.

Example 2:

Create a table for $4x - 2y = 10$ and use it to find the y-intercept and slope of the line.

x	y: $y = 2x - 5$	(x, y)
–2	$2(-2)-5=-9$	$(-2,-9)$
–1	$2(-1)-5=-7$	$(-1,-7)$
0	$2(0)-5=-5$	$(0,-5)$
1	$2(1)-5=-3$	$(1,-3)$
2	$2(2)-5=-1$	$(2,-1)$
3	$2(3)-5=1$	$(3,1)$

<u>Answer</u>:

To make a table, we will solve the equation for y:

$4x - 2y = 10$ subtract $4x$ from each side

$-2y = -4x + 10$ divide every term on both sides by -2

$y = 2x - 5$

We use the rule $y = 2x - 5$ to build a table.

The y-intercept has $x = 0$.

The point in the table that has $x = 0$ is $(0, -5)$.

To find the slope, select any two points: $(2, -1)$ and $(3, 1)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-(-1)}{3-2} = \frac{2}{1} = 2$$

Example 3:

Consider the linear equation: $f(x) = \frac{1}{2}x + 3$.

a) Graph the line by finding the horizontal and vertical axis intercepts (x- and y-intercepts), and plotting those two points.

b) Use the graph to find the vertical axis intercept and the slope of the line.

<u>Answers</u>:

a) For the linear equation $f(x) = \frac{1}{2}x + 3$,

To find the x-intercept, we let $f(x) = 0$.

$0 = \frac{1}{2}x + 3$ subtract 3 from both sides

$-3 = \frac{1}{2}x$ multiply both sides by 2

$-6 = x$ The x-intercept is $(-6, 0)$.

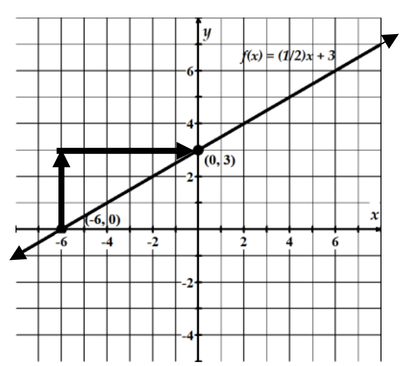
To find the y-intercept, we let $x = 0$.

$y = \frac{1}{2}(0) + 3$

$y = 3$

The y-intercept is $(0, 3)$.

b) The y-intercept is the point where the line crosses the y-axis: $(0, 3)$.
To find the slope, select two points and find $\frac{\text{rise}}{\text{run}}$. We can use the x- and y-intercept points: $(-6, 0)$ and $(0, 3)$. To go from $(-6, 0)$ to $(0, 3)$, we move up 3 spaces and over to the right 6. Therefore the slope is $\frac{3}{6} = \frac{1}{2}$.



Now let's summarize our results from Examples 1 – 3:

Example	Equation in general form	Equation solved for y	y-intercept (0, b)	Slope (m)
1	$3x+y=3$	$y=-3x+3$	$(0, 3)$	–3
2	$4x-2y=10$	$y=2x-5$	$(0, -5)$	2
3		$y=\frac{1}{2}x+3$	$(0, 3)$	$\frac{1}{2}$

Look closely at the columns for: **Equation solved for *y*, *y*-intercept (0, *b*), and Slope (*m*).** Can you find b and m when each equation is solved for y? This conclusion is so important, that we call an equation solved for y or $f(x)$ the slope-intercept form of an equation of a line.

The Slope-Intercept Form

The equation, $y=mx+b$ is the equation of any line that has a slope of m and a y-intercept of b. In function notation, the slope intercept form is $f(x)=mx+b$.

Writing an Equation of a Line in Slope-Intercept Form

Knowing how to write the equation of a line is essential. Every line has a name. One of the easiest ways to name a line is to write the equation in slope-intercept form.

To write the equation of a line in slope-intercept form, $y=mx+b$ or $f(x)=mx+b$, you need to know the slope and the y-intercept of the line. Then, replace the m with the slope of the line and the b with the b part of the y-intercept (0, b).

In Examples 1 – 3, you were given tables and graphs and asked to find the slope and y-intercept.

In Examples 4 – 7, you will be given a table and a graph, but now you will not know the rule. You will be asked to write the equation of the line.

Example 4:

The following table has points on a line. Find the equation of the line.

x	y	(x, y)
–1	–5	(–1, –5)
0	–1	(0, –1)
1	3	(1, 3)
2	7	(2, 7)

Answer:

The y-intercept has $x = 0$. The point in the table that has $x=0$ is (0, –1), so $b=-1$.

To find the slope, select two points. We can use (1, 3) and (2, 7).

$m=\frac{y_2-y_1}{x_2-x_1}=\frac{7-3}{2-1}=\frac{4}{1}=4$. Therefore the equation of this line is: $y=4x-1$.

Example 5:

The following table has points on a line. Find the equation of the line.

x	y	(x, y)
– 2	–1	(–2, –1)
0	0	(0, 0)
2	1	(2, 1)
4	2	(4, 2)

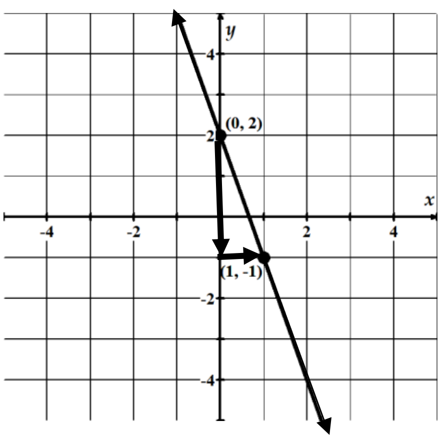
Answer:

The y-intercept has $x = 0$. The point in the table that has $x = 0$ is (0, 0), so $b = 0$.
To find the slope, select two points. We can use (0, 0) and (2, 1). $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-0}{2-0} = \frac{1}{2}$

The equation of this line is: $y = \frac{1}{2}x + 0$ *or* $y = \frac{1}{2}x$.

Example 6:

Find the equation of a line with the following graph.



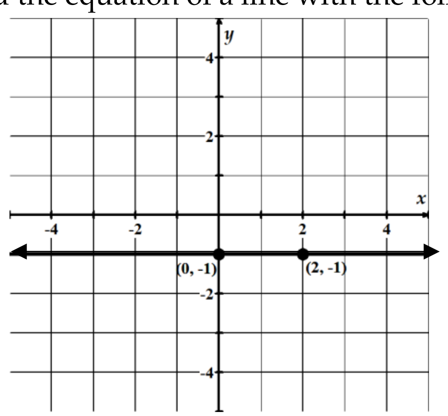
Answer:

The y-intercept is the point where the line crosses the y-axis: (0, 2), so $b = 2$.
To find the slope, select two integer points and find $\frac{\text{rise}}{\text{run}}$. We can use (0, 2) and (1, – 1). To go from (0, 2) to (1, – 1), we move down 3 spaces and over to the right 1. Therefore the slope is $\frac{-3}{1} = -3$

The equation of this line is: $y = -3x + 2$.

Example 7:

Find the equation of a line with the following graph.



Answer:

The y-intercept is the point where the line crosses the y-axis: (0, –1), so $b = -1$.
This is a horizontal line, so we know that the slope is zero. Therefore $m = 0$.
The equation of this line is: $y = (0)x - 1$ *or* $y = -1$.

Helpful Tip:

The equation of the horizontal line in the previous example reminds us of Section 2.7 (see Example 4 in Section 2.7), where we saw the following conclusions:
The equation of a horizontal line has the form: $y = b$.
The equation of a vertical line has the form: $x = a$.

Practice Exercises: Writing the equation of a line when given the table or graph

For exercises 1 – 6, find the slope and y-intercept for each table or graph. Then write an equation of each line.

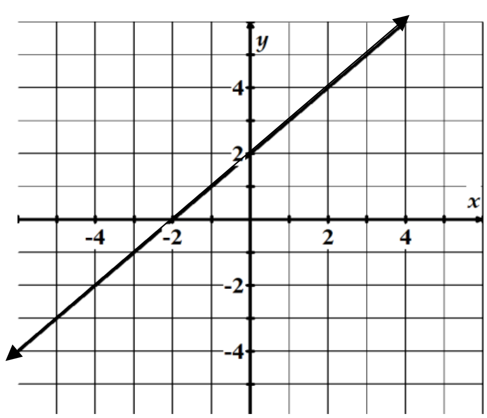
1)

x	y	(x, y)
–10	4	(–10, 4)
0	3	(0, 3)
10	2	(10, 2)
20	1	(20, 1)
30	0	(30, 0)

4)

x	y	(x, y)
–1	–12	(–1, –12)
0	–2	(0, –2)
1	8	(1, 8)
2	18	(2, 18)

2)



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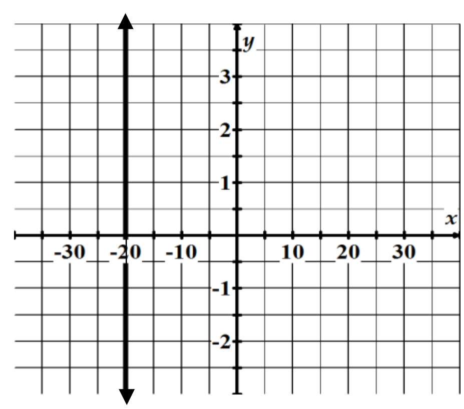

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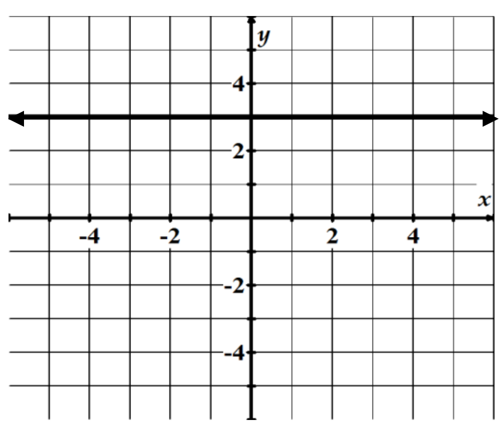
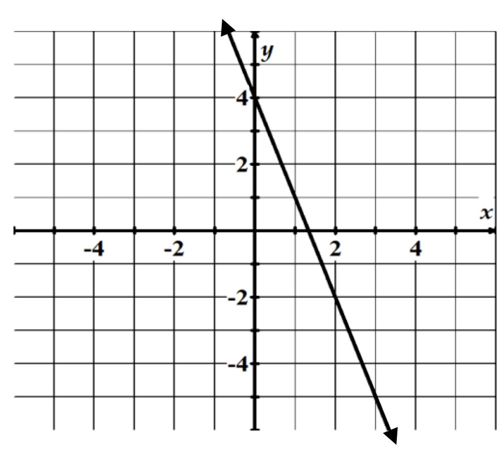


6)



We can also write an equation of a line if we are given the slope and the *y*-intercept, as in the next example.

Example 8:

Write the equation of each of the following lines in slope-intercept form.

a) $m = -1$, *y*-intercept (0, –1)

b) $m = -\frac{1}{3}$, *y*-intercept (0, 4)

Answers:

a) In the equation $y = mx + b$ we replace *m* with –1 and *b* with –1. The equation of this line is: $y = -1x - 1$ *or* $y = -x - 1$.

b) In the equation $y = mx + b$ we replace *m* with $-\frac{1}{3}$ and *b* with 4. The equation of this line is: $y = -\frac{1}{3}x + 4$.

Writing an Equation in Slope-Intercept Form, given the Slope and a Point on the Line

We know that to write an equation of a line in slope-intercept form, we need the slope and the *y*-intercept. But what if we are given the slope and a point on the line that is not the *y*-intercept? The next example addresses this situation.

Example 9:

Write the equation of each of the following lines in slope-intercept form.

a) $m = 2$, passes through (3, –1)

b) $m = -1$, passes through (3, –1)

Answers:

a) In the equation $y = mx + b$ we replace *m* with 2 and we need to find *b.* (We know that the point (3, –1) is not the *y*-intercept, since the value of *x* should be zero in the *y*-intercept.) We replace *m* with 2, *x* with the *x*-coordinate 3, and *y* with the *y*-coordinate –1. The equation becomes: $-1 = 2(3) + b$. We now simplify to $-1 = 6 + b$ and solve for *b.* Subtracting 6 from each side gives the result: $b = -7$. The equation is $y = 2x - 7$.

b) In the equation $y = mx + b$ we replace *m* with –1 and we need to find *b.* We replace *m* with –1, *x* with the *x*-coordinate 3, and *y* with the *y*-coordinate –1. The equation becomes: $-1 = -1(3) + b$ and we now simplify to $-1 = -3 + b$ and solve for *b.* Adding 3 to both sides gives the result: $b = 2$. The equation is $y = -x + 2$.

Writing an Equation in Slope-Intercept Form if given Two Points

Once again, to write an equation of a line in slope-intercept form, we need the slope and the y-intercept. So how can you write an equation of a line if you are only given two points on the line? To begin, you can use the slope formula and the two points to find the slope.

Example 10:

Write the equation of each of the following lines in slope-intercept form.

a) The line that passes through (2, – 1) and (5, – 4).

b) The line that passes through (– 3, – 1) and (– 3, 5).

Answers:

a) In the equation $y = mx + b$ we are not given m or b. However, since we are given two points, we can use the slope formula to find the slope of the line.
Identify x_1, y_1, x_2, and y_2:

x_1	y_1
2	–1

x_2	y_2
5	– 4

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4-(-1)}{5-(2)} = \frac{-3}{3} = -1$$

In the equation $y = mx + b$ we replace m with –1 and we need to find b. Pick one of the points, such as (2, – 1). We replace x with the x-coordinate 2, and y with the y-coordinate –1. The equation becomes: $-1 = -1(2) + b$. We then simplify to $-1 = -2 + b$ and solve for b. Adding 2 to both sides gives the result: $b = 1$.
Therefore the equation is $y = -x + 1$.

b) In the equation $y = mx + b$ we are not given m or b. However, since we are given two points, we can use the slope formula to find the slope of the line.
Identify x_1, y_1, x_2, and y_2:

x_1	y_1
–3	–1

x_2	y_2
–3	5

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-(-1)}{-3-(-3)} = \frac{6}{0}$$ which is undefined.

When the slope is undefined there is no slope, and the line is vertical. If we plot the two points, we can see that the line is vertical. The equation of a vertical line is $x = a$, and therefore the equation of this line is $x = –3$.

Helpful Tip:

All lines except a vertical line can be written in "y =" form. The equation of a vertical line is $x = a$.

Practice Exercises: Writing the equation of a line

For exercises 7 – 12, use the given information to write the equation of each line in slope-intercept form, if possible.

7) $m = -4$, passes through (–1, 1).

8) $m = 0$, passes through (–1, 1).

9) m is undefined, passes through (–1, 1).

10) The line that passes through (2, –1) and (5, – 4).

11) The line that passes through (2, –1) and (5, – 1).

12) The line that passes through (–4, – 1) and (6, 4).

Writing an Equation in Slope-Intercept Form if given the Line in General Form

If an equation is given in general form, $ax + by = c$, in order to find the slope and y-intercept you will need to solve for y.

Example 11:

Write the equation $x + 3y = 9$ in slope-intercept form and find the slope and y-intercept.

<u>Answer</u>:
To find the slope and y-intercept of the line $x + 3y = 9$ subtract x from each side and divide every term on both sides by 3. The result is: $3y = -x + 9$; or $y = \frac{-x}{3} + 3$. We can now identify the y-intercept as (0, 3) and the slope as $m = \frac{-1}{3}$.

Practice Exercises: Changing a line from general form to slope-intercept form

For exercises 13 – 14, write the equation of each line in slope-intercept form.

13) $3x - 2y = -4$

14) $5x - y = 1$

2.9 SUMMARY

1. The equation for a **linear function** can be written as $y = mx + b$ or $f(x) = mx + b$, where

- m is the ______________ of the line, and
- b is the __________________of the line, which is the point (____, ____)

Example: $3x - y = 4$ is a linear equation because it can be written as $y = 3x - 4$.

$m = 3$ is the _______ of the line

$b = -4$ gives the _________ of the line, which is the point ____.

2. The equation $y = mx + b$ is called **slope-intercept form** because you can quickly find the slope, ____, and the y-intercept, ____, from this form.

Example: Match the equation with its description.

a) Line with slope 4 passing through (0, –3).	$y = -3x - 4$
b) Line with $m = -3$, containing (0, – 4).	$y = -4x + 3$
c) Line with slope – 4 and y-intercept 3	$y = -3 + 4x$

3. To write a line in slope-intercept form, you must know the slope and the y-intercept.

Steps:

I. Find the slope, m.

II. Find b.

III. Write the equation of the line in slope-intercept form, $y = mx + b$.

Example: Use the table below to write an equation of the line.

x	y	(x, y)
–1	–12	$(-1, -12)$
0	–2	$(0, -2)$
1	8	$(1, 8)$
2	18	$(2, 18)$

I. To find the slope, select two points from the table, such as (1, 8) and (2, 18).

$$m = \frac{y_2 - y_1}{x_2 - x_1} =$$

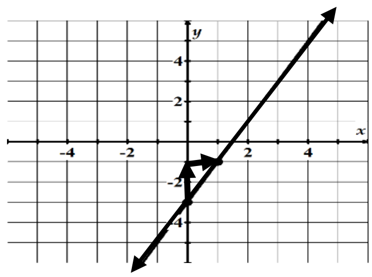
II. Find b.

The y-intercept has x =_______. The point in the table that has $x = 0$ is ________, so b =_______.

III. The equation of the line is

$y = \boxed{}\, x + \boxed{}$

Example: Use the graph below to write an equation of the line.

	**I.** To find the slope, select two integer points and find $\frac{\text{rise}}{\text{run}}$. We can use (0, –3) and (1, – 1). To go from (0, –3) to (1, –1), we move up _____ spaces and over to the right____________. Therefore the slope is ___.
II. Find b.	The y-intercept is the point where the line crosses the y-axis:___________, so b = _______.
III. The equation of the line is	$y = \square x + \square$

Example: Write the equation of the line with slope $m = -2$ that passes through (– 3, 5).

I. Find the slope	The slope is given: $m = -2$
II. Find b. • Start with $y = mx + b$ • Substitute the given point for x and y and the slope for m. Leave b as the variable. • Solve for b.	Find b. $y = -2x + b$ $5 = -2(-3) + b$
III. Write the equation of the line in $y = mx + b$ form, by substituting values for m and b, and leaving x and y as variables.	The equation of the line is $y = \square x + \square$

Example: Find the equation of the line that passes through the points (– 3, 5) and (6, 2).

I. Find the slope	The slope of this line is $m = \frac{2-5}{6-(-3)} =$
II. Find b. • Start with $y = mx + b$ • Substitute one of the given points for x and y and the slope for m. Leave b as the variable. • Solve for b.	Find b. $y = -\frac{1}{3}x + b$
III. Write the equation of the line in $y = mx + b$ form, by substituting values for m and b, leaving x and y as variables.	The equation of the line is $y = \square x + \square$

4a) Write an equation of a line in which the slope is zero, passing through (0, –3).________

b) Write an equation of a line that has no slope, passing through (0, –3). ____________________

Unit 3: Systems of Linear Equations, Statistics and Probability

3.1 Linear Functions: Graphing

When graphing a line we have been using two methods: we make a table of values and plot the resulting points, or we find the *x*- and *y*-intercepts and then plot and connect the two points. However, if we can identify some properties of the line such as the slope and one point on the line, we may be able to make a graph much more quickly and easily.

Graphing a Linear Equation given the Slope and a Point on the Line

To graph a linear equation given the slope, *m,* and a point on the line:

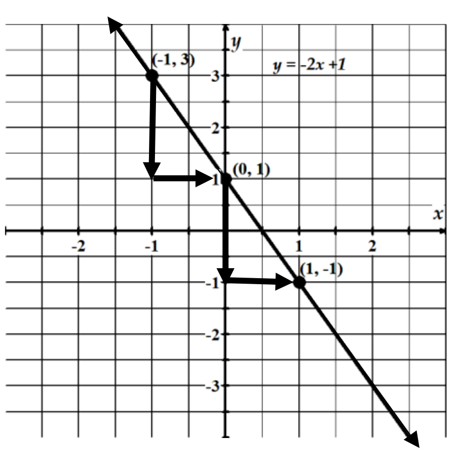
1) **Plot the given point.**
2) **Use the slope, $\frac{\text{rise}}{\text{run}}$, to determine another point on the line. To help draw a straight line, continue to use the slope to determine one or two additional points.**
3) **Draw a line through the points and extend the line.**
4) **Label the graph – write the equation of the line on the graph.**

Example 1:
Graph each of the following lines using the given slope and point. After graphing each line, use the graph to write the equation of the line in slope-intercept form: $y = mx + b$.

a) $m = -2$, passing through (–1, 3).

b) $m = \frac{1}{2}$, passing through (4, –2).

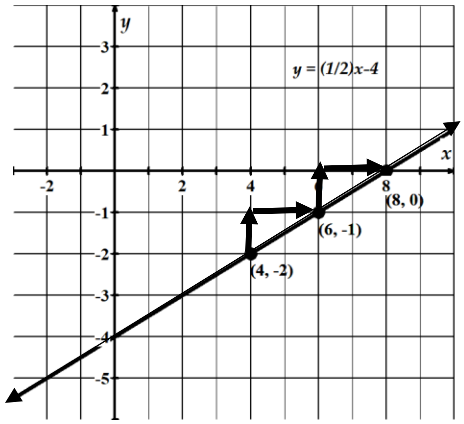
<u>Answers</u>:
a)



1) Plot the point (–1, 3).
2) Since the slope is $\frac{-2}{1}$, from the given point we move down 2 spaces and over to the right, 1, determining the points (0, 1) and (1, –1).
3) Connect the points and extend the line.

To write an equation of this line, we need the slope, *m*, and the *y*-intercept, *b,* (0, 1). The equation of this line is : $y = -2x + 1$.

b)



1) Plot the point (4, – 2).

2) Since the slope is $\frac{1}{2}$, from the given point we move up 1 space and over to the right, 2, determining the points (6, –1) and (8, 0).

3) Connect the points and extend the line.

To write an equation of this line, we need the slope, *m*. From the graph, we see that the *y*-intercept is (0, – 4). The equation of this line is : $y = \frac{1}{2}x - 4$.

Practice Exercises: Graphing a linear equation given the slope and a point on the line

For exercises 1 – 4, graph each of the following lines on graph paper, using the given slope and point. After graphing each line, use the graph to write the equation of the line in slope-intercept form: $y = mx + b$.

1) $m = -\frac{2}{3}$, passing through (3, 0).

2) $m = 2$, passing through (1, –1).

3) $m = 0$, passing through (–1, 3).

4) No slope, passing through (–1, 3).

Graphing a Linear Equation given the Slope and the *y*-Intercept

What if the given point is the *y*-intercept? If you are given the slope, *m*, and *y*-intercept (0, *b*), is that enough information to graph the line?

Example 2:

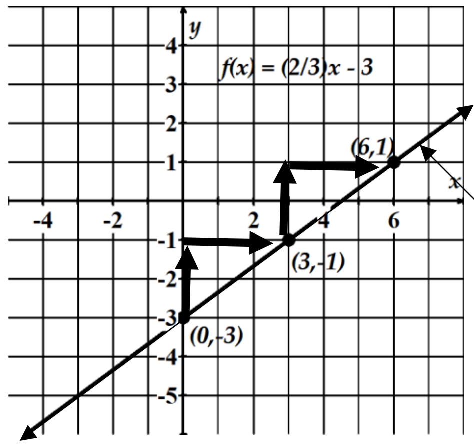
Graph each of the following lines using the given slope and point. After graphing each line, use the graph to write the equation of the line in slope-intercept form: $f(x) = mx + b$.

a) $m = \frac{2}{3}, \ b = -3$

b) $m = -\frac{2}{3}, \ b = -3$

Answers:

a)



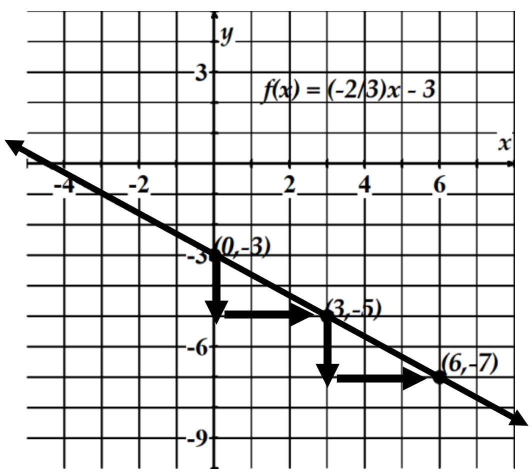
1) Plot the y-intercept point (0, –3).

2) Since the slope is $\frac{2}{3}$, from the given point we move up 2 spaces and over to the right, 3, determining the points (3, – 1) and (6, 1).

3) Connect the points and extend the line.

To write an equation of this line, we need the slope, m, and the y-intercept, b. The equation of this line is : $f(x)=\frac{2}{3}x-3$.

b)



1) Plot the y-intercept point (0, – 3).

2) Since the slope is $-\frac{2}{3}$, from the given point we move down 2 space and over to the right, 3, determining the points (3, – 5) and (6, – 7).

3) Connect the points and extend the line.

To write an equation of this line, we need the slope, m, and the y-intercept, b. The equation of this line is : $f(x)=-\frac{2}{3}x-3$.

Practice Exercises: Graphing a linear equation given the slope and the y-intercept

For exercises 5 – 8, graph each of the following lines on graph paper, using the given slope and y-intercept. After graphing each line, use the graph to write the equation of the line in slope-intercept form: $f(x)=mx+b$.

5) $m=-5,\ b=3$

6) $m=3,\ b=0$

7) $m=\frac{3}{2},\ b=-3$

8) $m=-\frac{1}{4},\ b=2$

Graphing a Line given the General Form of a Linear Equation

In Section 2.7, we learned that the linear equation $ax+by=c \quad (a \neq 0, \;\; b \neq 0)$ is said to be in general form. If you are asked to graph an equation in general form by using the y-intercept and slope, how would you begin?

Example 3:

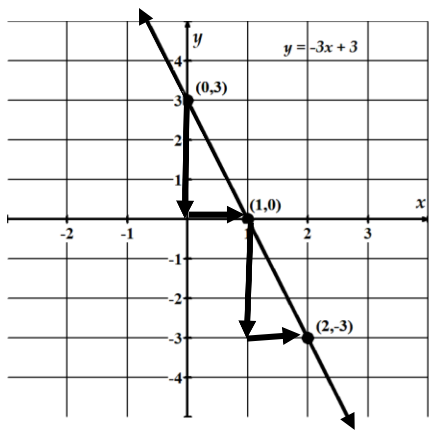
Write the equation of each of the following lines in slope-intercept form: $y=mx+b$. Then graph each line using the slope and y-intercept.

a) $3x+y=3$ **b)** $2x+3y=-6$

<u>Answers</u>:

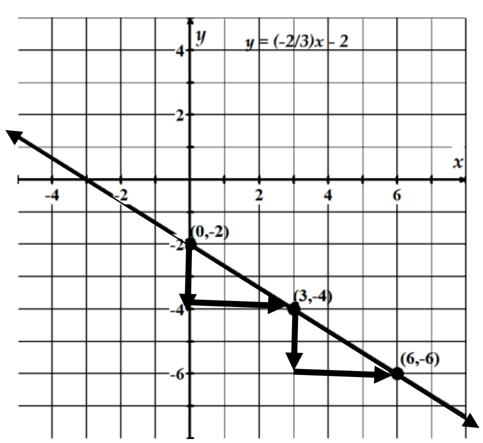
To graph a line using the slope and y-intercept, first we need to solve the equation for y so that the equation is in $y=mx+b$ form.

a)



1) First write the equation in $y=mx+b$ form:
   $3x+y=3$ subtract $3x$ from both sides.
   $y=\;-3x+3$ therefore $b=3, \quad m=\dfrac{-3}{1}$.
2) Plot the y-intercept point (0, 3).
3) Since the slope is $\frac{-3}{1}$, from the given point we move down 3 spaces and over to the right, 1, determining the points (1, 0) and (2, – 3).
4) Connect the points and extend the line.

b)



1) First write the equation in $y=mx+b$ form:
   $2x+3y=-6$ subtract $2x$ from both sides
   $y=\dfrac{-2}{3}x-2$ divide every term by 3
   therefore $b=-2, \quad m=\dfrac{-2}{3}$.
2) Plot the y-intercept point (0, – 2).
3) Since the slope is $\frac{-2}{3}$, from the given point we move down 2 spaces and over to the right, 3, determining the points (3, – 4) and (6, – 6) .
4) Connect the points and extend the line.

Practice Exercises: Graphing a linear equation given the general form of a linear equation

For exercises 9 – 12, write the equation of the line in slope-intercept form: $y = mx + b$. Then **graph each of the following lines on graph paper**, using the slope and y-intercept.

9) $3x + y = -6$

10) $2x - 5y = 10$

11) $2x - 3y = -3$

12) $4x - 6y = -12$

Parallel Lines

Lines in a coordinate plane are parallel if they never meet. The classic example of parallel lines is railroad tracks. As you examine the graph of two parallel lines in Example 4, look at their equations in slope-intercept form, and think about which part of each equation is the same.

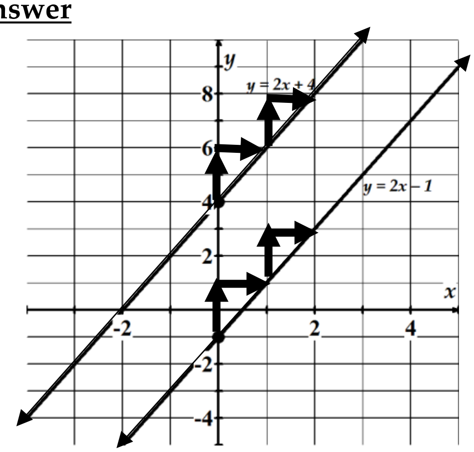
Example 4:

Graph each of the following lines on the same coordinate plane, using the slope and y-intercept.

a) $y = 2x + 4$

b) $y = 2x - 1$

__**Answer**__



Helpful Tip:

Parallel lines have the same slope. Lines that have the same slope with different y-intercepts are parallel. When a linear equation is written in $y = mx + b$ form, parallel lines have the same m but different y-intercepts.

Example 5:

Write the equation of a line parallel to $y = -2x+1$ that passes through the point (– 3, 4).

<u>Answer</u>:

To write an equation of a line, we need to know the slope (m) and the y-intercept (b). If we know that our new line is parallel to $y = -2x+1$, then we know the slope of the new line is also –2.

To find b for the new line, we substitute the slope for m and the given point for x and y.

$y = mx+b$ substitute: $m = -2$, $x = -3$, $y = 4$

$4 = -2(-3) + b$

$4 = 6 + b$

$-2 = b$

Therefore the equation of the new line is: $y = -2x-2$.

Practice Exercises: Parallel lines

For exercises 13 – 14, answer questions about parallel lines.

13) Write the equation of any line that is parallel to: $y = 4x-1$.

14) Circle the two lines that are parallel:

a) $y = -x-2$ **c)** $x-y = -2$

b) $y = -2x-2$ **d)** $x+y = 8$

For exercises 15 – 16, graph each pair of lines on the same coordinate plane and explain whether the two lines in each exercise are parallel. Use the slope of each line in your explanation.

15) $y = -2x+1$ and $2x+y = 3$

16) $y = -2x+1$ and $2x-y = 1$

For exercises 17 – 20, write the equation of a line described in each exercise.

17) parallel to $y = 3x+1$ that passes through the point (– 3, 4).

18) parallel to $2x-y = 1$ that passes through the point (– 3, 4).

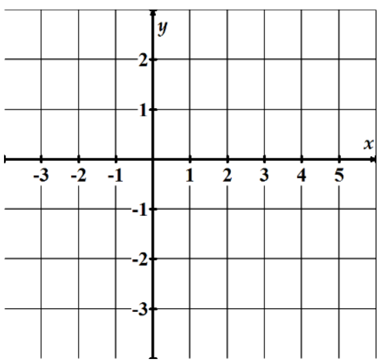
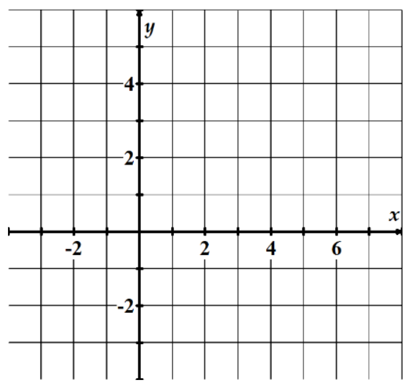
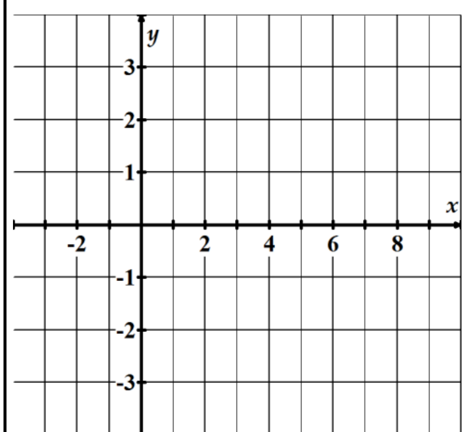
19) parallel to $-x+y = -2$ that passes through the point (0, 4).

20) parallel to $y = 8$ that passes through the point (– 3, 4).

3.1 SUMMARY

Graphing Lines

1. To graph a line, you need to find at least two points on the line.

If given the slope and a point, ➢ Start at the point and use the slope to find another point. Continue using the slope to get an additional point to help graph a straight line.	If given the equation in slope-intercept form, $y = mx + b$, ➢ Start at the point ________ and use the slope to find the next two points.	If given a line in any other form, ➢ Convert to slope-intercept form and proceed as at left. **OR** ➢ Graph by finding the *x*- and *y*-intercepts.
Graph the line passing through (–3, 2) with slope $m = -\dfrac{1}{4}$.  	Graph $y = \dfrac{2}{3}x - 1$.  	Graph $3x - 4y = 12$. 

2. Two lines are **parallel** if they have the same ______________ and different ______________ .
Example: $y = -3x + 6$ and $3x + y = -4$ are parallel because they both have slope $m =$ _____.

Example: Find the equation of the line parallel to $x - 5y = 15$ that passes through (0, 6).

IV. Find the slope. * Solve the given equation for *y* to find the slope. This will be the slope of the parallel line.	$x - 5y = 15$ Solve for *y*. $y = \square x + \square$ The slope is _____.
V. Find *b*. * In this example, the *y*-intercept, *b* is given.	Write *b*.
VI. Write the equation of the line in $y = mx + b$ form, by substituting values for *m* and *b*, leaving *x* and *y* as variables.	The equation of the line parallel to the given line is: $y = \square x + \square$

3.2 Linear Functions: Applications

In this section we will focus on how to use linear functions to solve real-life problems. When solving practical problems, it is often more convenient to introduce two variables rather than only one. However, two variables should be introduced only when two relationships can be found within the problem. If we can use the two variables to write an equation of a line, then we will be able to graph the line and have a visual understanding of the problem.

Identifying the Slope and y-intercept and Interpreting their Meaning

When working with a word problem that can be solved using a linear function, it is important to be able to interpret the meaning of the slope and y-intercept (also called the vertical axis intercept). Remember that the slope is a rate, and therefore should include units. The y-intercept is often referred to as the starting point, because it occurs when $x = 0$.

Example 1:

Identify the slope and vertical axis intercept in each of the following situations, and interpret their meaning. Include units with the slope.

a) An air conditioner repair contractor charges $75 to come to your home and an additional $50 an hour to do the job. The linear equation that represents this situation is: $y = 50x + 75$ where x represents the number of hours she works and y represents the total cost of the repair.

b) Clay received a $100 gift card to the B&N bookstore. Every few weeks he likes to go to that store and buy a car magazine that costs $8.50. The linear equation that represents this situation is: $f(x) = 100 - 8.5x$ where x represents the number of magazines, and $f(x)$ represents the value that remains on the gift card.

__Answers__:

a) The slope is 50. The cost of the repair increases at a rate of $50/hour. The y-intercept is (0, 75). It costs $75 for the visit to your home, even if the repair is not done (x hours = 0).

b) The slope is – 8.50. The value remaining on the gift card decreases at a rate of $8.50/magazine. The vertical axis intercept is (0, 100), because when it is new (0 magazines have been purchased), the value of the gift card is $100.

Practice Exercises: Interpreting the meaning of slope and the y-intercept in applications

For exercises 1 – 2, identify the slope and the y-intercept and interpret their meanings. Include units with slope.

1) Tuition at a local adult vocational school costs \$225 each semester in fees plus \$115 per credit. The function that represents this situation is: $f(x) = 115x + 225$ where x represents the number of credits taken in the fall semester, and y represents the total cost of tuition that semester.

2) A student is planning to take Statistics in Bermuda this summer, as part of Brookdale's International Studies Program. She has purchased a \$25 International calling card so that she can call home every day. A call from Bermuda to New Jersey costs \$3.25 per minute. The linear equation that represents this situation is: $y = 25 - 3.25x$ where x represents the number of minutes of phone calls, and y represents the value remaining on the card.

Translating an Application into a Linear Function

When translating an application with two variables into a linear function, try to use $y = mx + b$ $\left(\text{or } f(x) = mx + b\right)$ form to write the equation of a line. If you are given the rate of change, then that is the slope of the linear function so replace m with that rate of change. If you are given a starting point, then that is the vertical axis intercept, so replace b with that starting point value.

Example 2:

A major city offers a Metropass for \$50. A passenger with a Metropass can take any form of public transportation in that city for \$3 each trip.

a) Find the slope and interpret its meaning.
b) Find the vertical axis intercept and interpret its meaning.
c) Write an equation describing the output (the value remaining on the card) as a function of the input (the number of trips). Define the variables you use.

Answers:

a) The quantity that is changing in this application is a decrease of \$3 per trip. Therefore the slope is -3 and it means that the value remaining on the card decreases by \$3 for each trip taken.
b) The beginning value of the card is \$50. Therefore the vertical axis intercept is (0, 50) which means that the value of the card is \$50 when 0 trips are taken.
c) If we let x = number of trips taken and y = value remaining on the card, then the linear function representing this situation is: $f(x) = 50 - 3x$ or $y = 50 - 3x$.

Practice Exercises: Translating an application into a linear function

For exercises 3 – 4, identify the slope and the y-intercept and interpret their meanings. Then write an equation describing the output as a function of the input. Define the variables you use.

3) Carmine earns a weekly salary of $300 plus 8% of sales.

4) The town offers a prepaid card for residents who want to rent a tennis court. The cost of the card is $40, and the cost of renting a tennis court for each hour is $4.

Solving Applications using Linear Functions

To solve a word problem using linear functions:

1) Look for two relationships within the problem. Introduce two variables, one for each unknown quantity.

2) To form the linear function, decide what information is given: the slope and a point, two points, or the slope and y-intercept.

Example 3:

A skier got off a chairlift and began to ski down the mountain. He was descending at a constant rate of 150 feet per minute. After eight minutes, he had reached an altitude of 2300 feet.

a) Write the equation of the line that gives his altitude after x minutes of skiing down the mountain.

b) At what altitude did he get off the chairlift?

__Answers__:

a) ***Step 1:*** This word problem has two relationships: time (in minutes) and altitude (in feet). We are told to use x to represent minutes. Since we need to find the altitude, we will call the altitude in feet, y or $f(x)$.

Step 2: We will try to write the equation in $y = mx + b$ or $f(x) = mx + b$ form. We are given the rate, so we can replace m with -150 because he is going downhill. We are also told that: After eight minutes, he had reached an altitude of 2300 feet. We can write this information as an ordered pair. Since x is minutes and y is altitude, the ordered pair is (8, 2300). We can now write an equation given the slope (-150) and a point on the line (8, 2300).

$y = mx + b$ replace m, x, and y and solve for b

$$2300 = -150(8) + b$$

$$2300 = -1200 + b$$

$$3500 = b$$

Therefore the equation of this line is: $y = -150x + 3500$ or $f(x) = -150x + 3500$

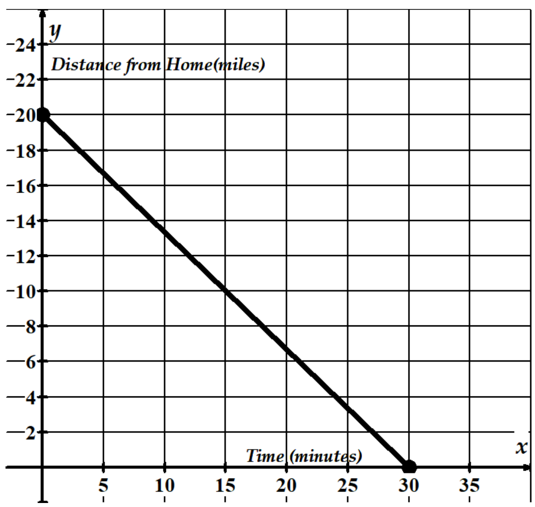
b) The skier got off the chairlift at time: $x = 0$ This is really asking for the y-intercept, which is 3500. Therefore he got off the chairlift at 3500 feet.

Helpful Tips:

Defining the variables is an extremely important part of solving an application. We can often find the two relationships and the unknown quantities by looking for the actual question. In Example 3, the question is: At what altitude did he get off the chairlift? This tells us we are trying to find the altitude, y. We were given the variable to use for minutes, x.

Example 4:

The graph below shows a person's distance from home as a function of time.



a) Identify the vertical axis intercept. Write it as an ordered pair and interpret its practical meaning.
b) Identify the horizontal axis intercept. Write it as an ordered pair and interpret its practical meaning.
c) Determine a linear equation to model this situation. Indicate what each variable represents.
d) How far is this person from home at three minutes?

Answers:

a) The vertical axis intercept is (0, 20) which means that when he first left his location (time $x = 0$) he was 20 miles from home.
b) The horizontal axis intercept is (30, 0) which means that after 30 minutes he was 0 miles from home; in other words it took him 30 minutes to get home.
c) In this problem, x = time (in minutes) and y = distance from home (in miles). To form the equation $y = mx + b$ we are given b, which is (0, 20). Since we are given two points, we can use the slope formula to find the slope of the line.
Identify $x_1, y_1, x_2,$ and y_2:

x_1	y_1
0	20

x_2	y_2
30	0

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-(20)}{30-(0)} = \frac{-20}{30} = \frac{-2}{3}$$

The equation of the line that models this situation is: $y = \frac{-2}{3}x + 20$ or $f(x) = -\frac{2}{3}x + 20$.

---

d) To find how far from home this person is at 3 minutes, we let $x = 3$.

$$y = \frac{-2}{3}(3) + 20$$
$$y = -2 + 20$$
$$y = 18$$

At 3 minutes, this person is still 18 miles from home.

Practice Exercises: Solving applications using linear functions

For exercise 5, answer the following questions. Define the variables you use.

5) A candy company has a machine that produces candy canes. The number of candy canes produced depends on the amount of time the machine has been operating. The machine produces 160 candy canes in five minutes. In twenty minutes, the machine can produce 640 candy canes.

a) Define the input variable and the output variable. Write the given information as two ordered pairs.

b) Use the two points to form a linear equation that models this situation.

c) Determine the vertical axis intercept of this linear equation. Write it as an ordered pair and interpret its practical meaning.

d) Determine the horizontal axis intercept of this linear equation. Write it as an ordered pair and interpret its practical meaning.

e) How many candy canes will this machine produce in 1 minute?

f) How many candy canes will this machine produce in 1 hour?

g) Graph this linear equation **on graph paper**. Think about an appropriate scale.

3.2 SUMMARY

Applications of Linear Functions

1. A linear function can often be used to model applications that contain _______ quantities.

2. When forming a linear function that models an application, begin by trying to use the

_________________________________ form of a linear equation.

3. When translating words from an application, the slope is a ___________________ which can usually be identified by words such as miles per hour or dollars per credit.

4. To find the vertical axis intercept of a linear function, let _____________ = 0.

5. When forming a linear function from an application, the vertical axis intercept is often thought of as the _________________ point.

6. <u>**Example**</u>:

A student is depositing $55 each month into her savings account. If after 6 months her account contains $830, what was her initial deposit?

a) The input variable, x, represents___________________________.

b) The output variable, y, represents__________________________.

c) Find the slope and interpret its meaning. ____

d) To write an equation of the line in $y = mx + b$, replace ____________ with 55.

e) Write the ordered pair that is given in this problem.______________________

f) Find the equation of the line that models this situation.

g) Use the equation of the line to find the initial deposit.

h) What is another name for the initial deposit?____________________________________

3.3 Solving Systems of Equations using Graphs

We have solved equations in one variable such as $3x-4=11$ by adding 4 to both sides and then dividing by 3 (the only solution is $x = 5$). We also have methods to solve equations with more than one variable. For example, we solved a linear equation with two variables such as: $y=3x-4$ by looking at a table and graph. In fact when we graph that function, we connect two points with a straight line and extend the line, which indicates that there are an infinite number of solutions.

It turns out that to get one solution when working with linear functions in two variables, we need to have two equations. **When we are solving two or more equations, we call the equations a system of equations.** The brace { is often used to denote that the two equations occur together (simultaneously). When solving a system of equations, we are looking for an ordered pair (x, y) that makes both equations true.

The Solution to a System of Linear Equations

An ordered pair that makes both equations true in a system of linear equations is called the **solution to the system of equations**.

Example 1:

Show that the ordered pair (3, 10) is the solution to the system: $\begin{cases} y=3x+1 \\ y=2x+4 \end{cases}$.

<u>Answer</u>:
The ordered pair (3, 10) is the solution to the system if it is a solution in each equation.

$y=3x+1$ We let $x = 3$ and $y = 10$,	$y=2x+4$ We let $x = 3$ and $y = 10$,
$10 \stackrel{?}{=} 3(3)+1$	$10 \stackrel{?}{=} 2(3)+4$
$10=10$ √ true	$10=10$ √ true

Because we found that substituting (3, 10) in each equation resulted in a true statement, we can state that (3, 10) is the solution to the system.

Practice Exercises: The solution to a system of linear equations.

For exercise 1, show that the given ordered pair is a solution to the system.

1) Show that the ordered pair (2, 1) is the solution to the system: $\begin{cases} 3x-y=5 \\ x+y=3 \end{cases}$.

Solving a System of Linear Equations by Graphing

If the graph of a line is a picture of all the solutions, we can graph two lines on the same coordinate plane to see the solutions of both equations. We are interested in the point that is the solution for both lines. This would be where the lines intersect! If we can find the point of intersection of the lines then we have found the solution, which is the point that makes both equations true.

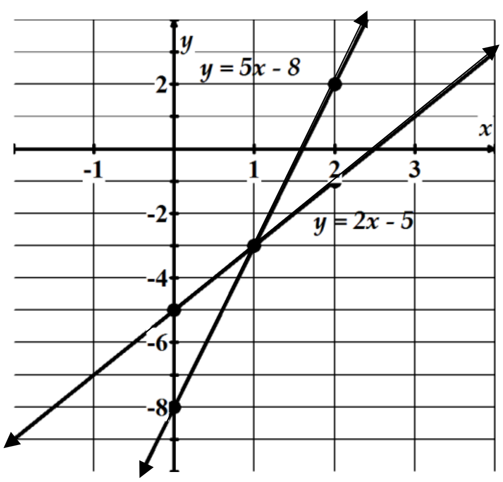
Example 2:

Graph each of the following lines on the same coordinate plane: $\begin{cases} y = 2x - 5 \\ y = 5x - 8 \end{cases}$ and find the solution.

<u>Answer</u>:

To graph $y = 2x - 5$ begin by placing a point on the y-intercept, (0, – 5). From that point, use the slope, $\frac{2}{1}$ to move up 2, over to the right 1, determining the points (1, – 3) and (2, – 1). Connect the points and extend the line in both directions. Be sure to label the line with its equation.

To graph $y = 5x - 8$ begin by placing a point on the y-intercept, (0, – 8). From that point, use the slope, $\frac{5}{1}$ to move up 5, over to the right 1, determining the points (1, – 3) and (2, 2). Connect the points and extend the line in both directions.



From the graph the point (1, – 3) is the solution.

To check, we substitute (1, – 3) into each equation:

$y = 2x - 5$	$y = 5x - 8$
$-3 \stackrel{?}{=} 2(1) - 5$	$-3 \stackrel{?}{=} 5(1) - 8$
$-3 = -3 \surd$	$-3 = -3 \surd$

The result is true for each equation. Therefore (1, – 3) is the solution to this system.

When two equations intersect so that the system of equations has a solution, this can help visualize some applications and even find a solution, as in Example 3.

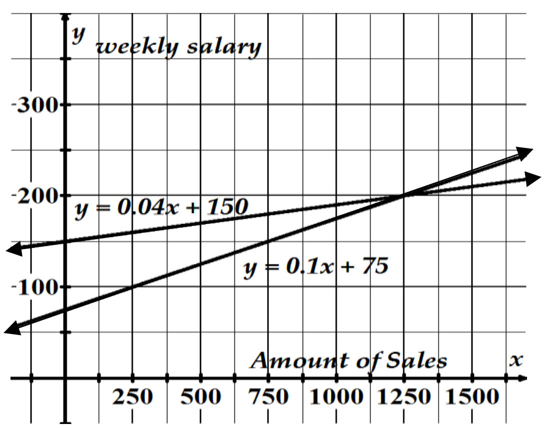
Example 3:

A student has been offered two jobs in sales, and she needs to decide which job to take. Job A will pay her \$150 each week plus 4% of sales. Job B will pay her \$75 each week plus 10% of sales.

a) Define variables that model the salary for Jobs A and B, as a function of sales.
b) Find the slope for Job A and the slope for Job B, and explain the meaning of each.
c) Find the vertical axis intercept for Job A and for Job B and explain the meaning of each.
d) Write a system of equations that models the Job A and Job B salaries.
e) Graph the system of equations. Be sure to use a good scale that will show the point of intersection.
f) Find a point for which the Job A salary is the same as the Job B salary.
g) How would you advise a friend which job to take?

<u>**Answers**</u>**:**

a) Let x = amount of sales and y = the weekly salary.
b) Job A: the salary increases at a rate of 4% of sales, so the slope is 0.04.
Job B: the salary increases at a rate of 10% of sales, so the slope is 0.1.
c) Job A: the vertical axis intercept is (0, 150) which means that in a week with no sales, her salary will be \$150.
Job B: the vertical axis intercept is (0, 75) which means that in a week with no sales, her salary will be \$75.
d) The system of equations that models the Job A and Job B salaries is: $\begin{cases} y = 0.04x + 150 \\ y = 0.1x + 75 \end{cases}$
e)



f) The point of intersection is (1250, 200) which means that if she sells \$1250, her weekly salary will be \$200 at each job.
g) If she sells less than \$1250 each week, then Job A will pay more. However, if her sales are more than \$1250 each week, then Job B will pay more.

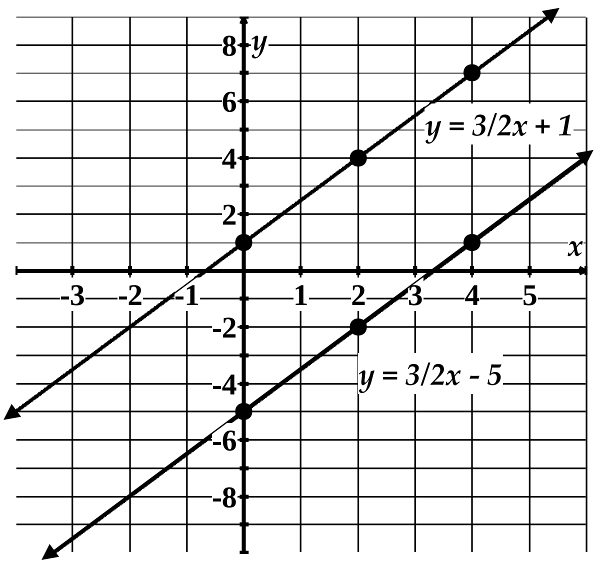
When we graph two linear equations on the same coordinate plane, will they always intersect? We examine this question in Examples 4 and 5.

Example 4:

Graph each of the following lines on the same coordinate plane: $\begin{cases} y = \dfrac{3}{2}x - 5 \\ y = \dfrac{3}{2}x + 1 \end{cases}$

and find the solution.

<u>Answer</u>:



To graph each equation, we start at the y-intercept and use the slope $\dfrac{\text{rise}}{\text{run}}$ to get to the next point. Then connect the points.

The two lines do not intersect! They are parallel. They have the same slope but different y-intercepts. If the lines do not intersect we know that there is no point that works in both equations. **Therefore there is <u>no solution</u> to this system of equations.**

Helpful Tip:

Since both lines are given in slope-intercept form, we could have noticed that both lines have the same slope. Remembering that parallel lines have the same slope and different y-intercepts, we would have known there was no solution without graphing the lines.

But be careful – **read the directions carefully**. If the directions ask you to graph the lines before solving the system, then you must graph each line.

Example 5:

Solve the following system of equations by graphing: $\begin{cases} 2x-6y=12 \\ 3x-9y=18 \end{cases}$

<u>Answer</u>:
To graph this system, we will need to solve each equation for y so that they are in $y=mx+b$ form.

$2x-6y=12$ subtract $2x$ from both sides

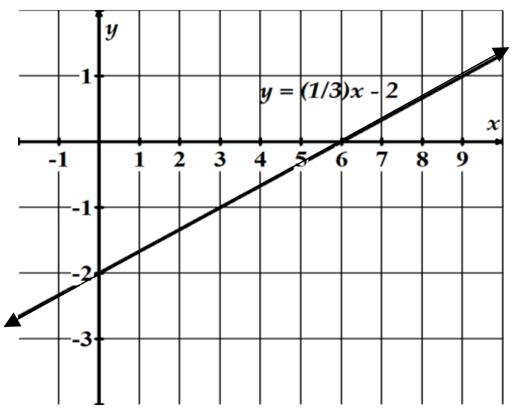
$-6y=-2x+12$ divide every term on both sides by -6

$y=\frac{1}{3}x-2$

$3x-9y=18$ subtract $3x$ from both sides

$-9y=-3x+18$ divide every term on both sides by -9

$y=\frac{1}{3}x-2$



Both equations are the same line! They have the same slope and the same y-intercept. As one line is directly on top of the other line, we can say that the lines "intersect" at all the points!

Here we say we have **<u>infinite solutions</u>**.

Helpful Tip:
Once we write both equations in slope-intercept form, if both equations are the same (with the same slope and same y-intercept) and if we are only asked to find the solution to the system (and not graph the system), we can state that there are infinite solutions without having to go through the work of graphing the equations.

Coincident Lines

When we graph a system, if both lines are exactly the same (with the same slope and same y-intercept), we say that the two lines **coincide** (one line fits exactly on top of the other). This means both lines have the same slope and also the same y-intercept. Therefore the system is **coincident**.

Practice Exercises: Solving systems of equations using graphs

For exercises 2 – 10,

i. Find the slope and y-intercept for each equation.
ii. Sketch the graphs for each system on the same coordinate plane.
iii. Solve each system based on graphs from ii.
iv. If the lines intersect, check the solution in each equation.

2) $\begin{cases} 2x - y = 1 \\ x + y = 8 \end{cases}$

3) $\begin{cases} 2x + y = 5 \\ x + y = 2 \end{cases}$

4) $\begin{cases} -x + y = -1 \\ y = x + 2 \end{cases}$

5) $\begin{cases} -2x + 3y = -2 \\ -6x + 9y = -6 \end{cases}$

6) $\begin{cases} 2x + y = 1 \\ -x + y = -5 \end{cases}$

7) $\begin{cases} 3x + 5y = 15 \\ 9x + 15y = 15 \end{cases}$

8) $\begin{cases} y = -3 \\ x + 2y = -4 \end{cases}$

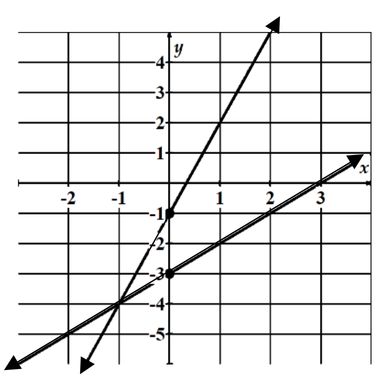
9) $\begin{cases} -3x + y = 5 \\ -x + y = 3 \end{cases}$

10) $\begin{cases} 2x + 3y = 6 \\ 3x + 4y = 6 \end{cases}$

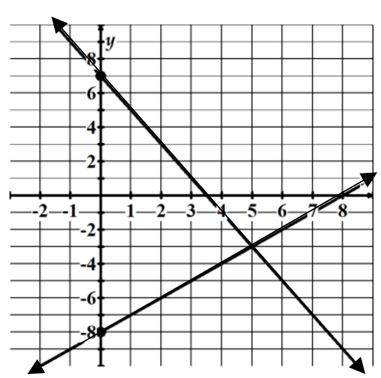
For exercises 11 – 12,

i. Write the equations in slope-intercept form.
ii. Write the system of equations that is represented by each graph.
iii. Find the point of intersection.
iv. Show that the point of intersection is the solution to the system.

11)



12)



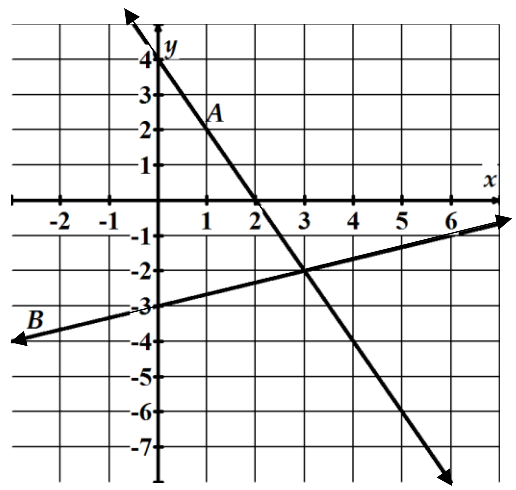
3.3 SUMMARY

In this class, we have studied systems of linear equations consisting of two equations in two variables.

1. To solve a system of linear equations graphically, graph the two ________ that represent the equations, and find the point of _______________. This is the solution to the system.

2. A system of linear equations can have ____, ____, or an ____________ number of solutions.

Example:



The equation of Line A is:

____

The equation of Line B is:

____

The system of equations in this graph is:

The solution to the system is: ____________________

Show the checks:

Line A:

Line B:

3.4 Solving Systems of Equations by Substitution

In the previous section we solved a system of linear equations by graphing. However, that process has limitations. First, it requires the graph to be perfectly drawn; if the lines are not straight we may arrive at the wrong answer. Second, graphing is not an effective method to use if the answer is really large, over 100 for example, or if the point of intersection includes a coordinate with fractions or decimals. For these reasons we will rarely use graphing to solve our systems. Instead, an algebraic approach will be used.

The first algebraic approach is called substitution. We will build the concepts of substitution through several examples, then end with a six-step process to solve problems using this method.

Solving a System of Linear Equations using Substitution

Example 1:

Solve the following system of equations using substitution: $\begin{cases} x = 5 \\ y = 2x - 3 \end{cases}$. Check the solution.

Answer:

To find the solution (x, y), we already know $x = 5$. Substitute this value into the other equation: $y = 2(5) - 3$. Evaluate, and the result is $y = 7$. We now have the solution (5, 7). Checking this, we have: $7 \stackrel{?}{=} 2(5) - 3,\ 7 = 7$ which is true. Therefore the solution is: (5, 7).

Helpful Tip:

When we know what one variable equals, we can plug that value (or expression) in for the variable in the other equation. **It is very important that when we substitute, we must place the substituted value in parentheses.**

Example 2:

Solve the following system of equations using substitution: $\begin{cases} 2x + 3y = 8 \\ y = -2 \end{cases}$. Check the solution.

Answer:

To find the solution (x, y), we already know that $y = -2$. Substitute this into the other equation $2x + 3(-2) = 8$ and solve for x. Add 6 to both sides and divide by 2. The result is $x = 7$. We now have the solution: (7, – 2). Checking this, we have: $2(7) + 3(-2) \stackrel{?}{=} 8,\ \ 8 = 8$ which is true.

Practice Exercises: Solving systems of equations by substitution

For exercises 1 – 2, solve each system by substitution and check each solution.

1) $\begin{cases} 3x-2y=10 \\ x=-1 \end{cases}$ **2)** $\begin{cases} 2x+3y=8 \\ y=5 \end{cases}$

Example 3:

Solve the following system of equations using substitution: $\begin{cases} 2x+3y=8 \\ y=2x \end{cases}$. Check the solution.

<u>Answer</u>:

To find the solution (x, y), we already know that $y=2x$. Substitute this into the other equation $2x+3(2x)=8$ and solve for x:

$$\begin{aligned} &2x+3(2x)=8 \\ &2x+6x=8 \\ &8x=8 \\ &x=1 \end{aligned}$$

The result is $x=1$ but that is not the final solution. We now need to find the value of y. Since $y=2x$ we can replace x with 1, resulting in: $y=2(1)$, or $y=2$. The solution is: (1, 2).

Finally, **check** that this ordered pair makes each equation true.

$2x+3y=8$	$y=2x$	
$2(1)+3(2)\stackrel{?}{=}8$	$2\stackrel{?}{=}2(1)$	Therefore the solution is: (1, 2).
$8=8$ √ true	$2=2$ √ true	

Substitution that requires using the Distributive Property

When substituting an expression that has terms connected by + or –, we will use the distributive property, as in Example 4.

Example 4:

Solve the following system of equations using substitution: $\begin{cases} 2x-3y=8 \\ y=x+2 \end{cases}$. Check the solution.

Answer:

To find the solution (x, y), we already know that $y = x+2$. Substitute this expression for y in the other equation: $2x-3(x+2)=8$ and solve for x:

$2x-3(x+2)=8$ apply the distributive property

$2x-3x-6=8$ combine like terms on the same side

$-x-6=8$ add 6 to both sides

$-x=14$ divide both sides by -1

$x=-14$

The result is $x=-14$ but that is not the final solution. We now need to find the value of y. Since $y=x+2$ we can replace x with –14, resulting in: $y=-14+2$, or $y=-12$.

The solution is: (–14, –12).

Finally, **check** that this ordered pair makes each equation true.

$2x-3y=8$	$y=x+2$	
$2(-14)-3(-12)\stackrel{?}{=}8$	$-12\stackrel{?}{=}-14+2$	Therefore the solution is: (–14, –12).
$8=8$ √ true	$-12=-12$ √ true	

Practice Exercises: Substitution that requires using the distributive property

For exercises 3 – 4, solve each system by substitution. **Check** each ordered pair makes each equation true.

3) $\begin{cases} 2x-3y=-1 \\ y=3x-2 \end{cases}$

4) $\begin{cases} 2x-3y=7 \\ x=y-2 \end{cases}$

Solving One of the Equations for a Variable

In Examples 1 – 4, each system had one equation that was solved for one variable, that is, one equation was written as $y=$ or $x=$. However, when we are given a system to solve, a variable with a coefficient of 1 is not always alone on one side of the equation. If this happens we take one of the equations and isolate a variable by solving for it, as in Example 5.

Example 5:

Solve the following system of equations using substitution: $\begin{cases} 3x+2y=1 \\ x-5y=6 \end{cases}$

Answer:
The variable with a coefficient of 1 is x in the second equation, so we will solve the second equation for x.

$x-5y=6$ add $5y$ to both sides

$x=5y+6$

Substitute this into the untouched equation: $3x+2y=1$. Replacing x with $5y+6$, we have

$3(5y+6)+2y=1$ distribute the 3

$15y+18+2y=1$ combine like terms on the same side

$17y+18=1$ subtract 18 from both sides

$17y=-17$ divide both sides by 17

$y=-1$

The result is $y=-1$ but that is not the final solution. We now need to find the value of x. Since $x=5y+6$, we can replace y with –1, resulting in: $x=5(-1)+6$, or $x=1$. The solution is: (1, – 1). Finally, **check** that this ordered pair makes each original equation true.

Example 6:

Solve the following system of equations using substitution: $\begin{cases} 4x-2y=2 \\ 2x+y=-5 \end{cases}$

Answer:
The variable with a coefficient of 1 is y in the second equation, so we will solve the second equation for y.

$2x+y=-5$ subtract $2x$ from both sides

$y=-2x-5$

Substitute this into the untouched equation: $4x-2y=2$. Replacing y with $-2x-5$, we have

$4x-2(-2x-5)=2$ distsribute the -2

$4x+4x+10=2$ combine like terms on the same side

$8x+10=2$ subtract 10 from both sides

$8x=-8$ divide both sides by 8

$x=-1$

The result is $x=-1$ but that is not the final solution. We now need to find the value of y. Since $y=-2x-5$ we can replace x with –1, resulting in: $y=-2(-1)-5$, or $y=-3$.
The solution is: (–1, –3). C**heck** that this ordered pair makes each original equation true.

Steps for Solving a System of Linear Equations by Substitution

We can summarize the process for solving a system of linear equations by substitution:

1) **Isolate one variable (preferably with a coefficient of 1) in one of the equations, so that it is written as *y* = or *x* =.**
2) **Substitute the expression into the untouched equation.**
3) **Solve that equation for one variable.**
4) **Substitute that value into one of the equations to find the value of the other variable.**
5) **Write the solution as an ordered pair.**
6) **Check that the solution makes each original equation true.**

Example 7:

Solve the following system of equations using substitution: $\begin{cases} 4x+3y=-14 \\ 2x+y=-8 \end{cases}$

<u>Answer</u>:

We can apply the steps for solving a system of linear equations by substitution:

1) The variable with a coefficient of 1 is y in the second equation. To solve that equation for y, we subtract $2x$ from both sides, resulting in $y=-2x-8$
2) Substitute $-2x-8$ for y in the untouched equation: $4x+3y=-14$
3) Solve the untouched equation for *x:*

   $4x+3(-2x-8)=-14$ distribute the 3

   $4x-6x-24=-14$ combine like terms on the same side

   $-2x-24=-14$ add 24 to both sides and divide by -2

   $-2x=10$

   $x=-5$
4) Find the other variable:

   $y=-2x-8$

   $y=-2(-5)-8$

   $y=2$
5) The solution to this system is $(-5,\ 2)$.
6) Check the solution in the original equations:

   $4x+3y=-14$

   $4(-5)+3(2)\stackrel{?}{=}-14$

   $-20+6\stackrel{?}{=}-14$

   $-14 - -14$ √ true

   $2x+y=-8$

   $2(-5)+2\stackrel{?}{=}-8$

   $-10+2\stackrel{?}{=}-8$

   $-8=-8$ √ true

If a system of linear equations has more than one variable with a coefficient of 1, then we can solve one equation for any of those variables, as in Example 8.

Example 8:

Solve the following system of equations using substitution: $\begin{cases} x+y=5 \\ x-y=-1 \end{cases}$

<u>Answer</u>:

The variables with a coefficient of 1 are x and y in the first equation, and x in the second equation. We will chose to solve for x in the first equation.

$x+y=5$ subtract y from each side

$x=-y+5$

Substitute $-y+5$ for x in the untouched equation $x-y=-1$:

$x-y=-1$

$(-y+5)-y=-1$ combine like terms and subtract 5 from each side

$-2y=-6$

$y=3$

The result is $y=3$ but that is not the final solution. We now need to find the value of x. Since $x=-y+5$ we can replace y with 3, resulting in: –(3) + 5, or $x = 2$. The solution is: (2, 3). Finally, **check** that this ordered pair makes each original equation true.

Solving Systems with Two Expressions for the Same Variable

Many applications can be modeled by a system of equations that has two different expressions for the same variable, as you will see on Lab 3.4. This type of system is also in the next example.

Example 9:

Solve the following system of equations using substitution: $\begin{cases} y=3x-2 \\ y=10x+5 \end{cases}$

Answer:

We already know from the first equation, that $y = 3x - 2$. Therefore, we can substitute $3x - 2$ for y in the untouched (second) equation $y = 10x + 5$:

$3x - 2 = 10x + 5$ subtract $3x$ from both sides

$-2 = 7x + 5$ subtract 5 from each side

$-7 = 7x$ divide both sides by 7

$x = -1$

The result is $x = -1$ but that is not the final solution. We now need to find the value of y. Since $y = 3x - 2$ we can replace x with –1, resulting in: $y = 3(-1) - 2$, or $y = -5$.
The solution is: (–1, –5). Finally, **check** that this ordered pair makes each original equation true.

Solving Systems with Infinite Solutions or No Solution

Just as with graphing, it is possible to have infinite solutions (same line) or no solution (parallel lines) with the substitution method. While we won't have a graph to look at and conclude if it is one or the other, the process takes an interesting turn as shown in the following examples.

Example 10:

Solve the following system of equations using substitution: $\begin{cases} y + 4 = 3x \\ 2y - 6x = -8 \end{cases}$

Answer:

The variable with a coefficient of 1 is y in the first equation, so we will solve the first equation for y.

$y + 4 = 3x$ subtract 4 from both sides

$y = 3x - 4$

Substitute this into the untouched equation: $2y - 6x = -8$. Replacing y with $3x - 4$, we have

$2(3x - 4) - 6x = -8$ distribute the 2

$6x - 8 - 6x = -8$ combine like terms on the same side

$-8 = -8$ which is true

The variables are gone! If the result is a true statement with no variables, we know that anything that works in the first equation will also work in the second equation. Therefore there are infinite solutions. If we write each line in $y = mx + b$ form, we may notice that the lines have the same slopes and same y-intercepts. This is a coincident system.

Example 11:

Solve the following system of equations using substitution: $\begin{cases} 6x-3y=-9 \\ -2x+y=5 \end{cases}$

<u>Answer</u>:

The variable with a coefficient of 1 is y in the second equation, so we will solve the second equation for y.

$-2x+y=5$ add $2x$ to both sides
$y=2x+5$

Substitute this into the untouched equation: $6x-3y=-9$. Replacing y with $2x+5$, we have

$6x-3(2x+5)=-9$ distribute the -3
$6x-6x-15=-9$ combine like terms on the same side
$-15=-9$ which is not true!

The variables are gone but the result is a false statement. Therefore there is no solution. Because we had a false statement and no variables, we know that no values will work in both equations.

We can check by writing each equation in $y=mx+b$ form. We can then see that the slopes are the same and y-intercepts are different. These two lines are parallel.

Helpful Tip:

When using substitution if the variables are gone and:

1) the result is a true statement, then the system has coincident lines and has infinite solutions.
2) the result is a false statement, then there is no solution. The system has parallel lines.

What if Neither Equation has a Variable with a coefficient of 1?

One more question needs to be considered: what if <u>neither equation</u> in the system has a variable with a coefficient of 1, as in solving the system: $\begin{cases} 5x-6y=-14 \\ -2x+4y=12 \end{cases}$

Solving this system by substitution would result in using fractions. The system is best solved using the second algebraic technique, which we will see in the next section.

Practice Exercises: Solving systems of equations by substitution

For exercises 5 – 14, solve each system of equations using substitution. Check each solution.

5) $\begin{cases} 2x-3y=8 \\ y=2 \end{cases}$

6) $\begin{cases} 3x-3y=6 \\ x=-1 \end{cases}$

7) $\begin{cases} x+3y=14 \\ y=2x \end{cases}$

8) $\begin{cases} 2x+3y=14 \\ 3x+y=7 \end{cases}$

9) $\begin{cases} 5x-8y=18 \\ 4x+y=7 \end{cases}$

10) $\begin{cases} 2x-y=1 \\ 4x-2y=4 \end{cases}$

11) $\begin{cases} 4x+8y=8 \\ x+2y=2 \end{cases}$

12) $\begin{cases} 3x+5y=9 \\ x=4y-14 \end{cases}$

13) $\begin{cases} y=3x-1 \\ y=2x+4 \end{cases}$

14) $\begin{cases} x=8-3y \\ x=3y-4 \end{cases}$

3.4 SUMMARY

To solve a system of equations by substitution, one of the equations must be solved for one of the variables. If one of the equations is not solved for a variable, look for a variable that has a coefficient of 1 or -1. This makes solving for that variable easier.

Example: $\begin{cases} 2x-3y=2 \\ x-2y=-4 \end{cases}$

WORK	STEP
	1. Solve one equation for a variable. **For this system, it will be easiest to solve the second equation for x. To get x to stand alone on one side, simply add ______ to both sides of the equation.**
	2. Substitute the expression you got in step 1 for the x in the untouched (first) equation. The expression should be placed in parentheses, as distribution will most likely be used.
	3. The result is that the first equation now has one variable. Solve this equation for y.
	4. Substitute the solution for y into one of the original equations, and solve for x.
	5. Write the solution either as an ordered pair, (x, y) or in "x = ___, y = ___" form.
	6. Check the solution in both of the original equations.

Interpreting Results From the Substitution Method

- There is ***one solution*** (x, y) to the system when a value can determined for both x and y.
- There are ***infinite solutions*** to the system when the variables are gone and a true statement remains (for example, $5 = 5$).
- There are ***no solutions*** to the system when the variables are gone and a false statement remains (for example, $2 = 4$).

Example: Solve the following system of equations using substitution: $\begin{cases} -8x-2y=1 \\ 4x+y=2 \end{cases}$

3.5 Solving Systems of Equations by Elimination

We have found that graphing is very limited when solving systems of equations. We then considered an algebraic method known as substitution. This is probably the most used technique in solving systems in various areas of algebra. However, substitution can get messy if we don't have a variable with a coefficient of 1 or -1. This leads us to our second algebraic method for solving systems of equations, which is known as either Elimination or Addition. We will set up the process in the following examples, and then define the five step process for solving by elimination.

Example 1:

Solve the following system of equations by elimination: $\begin{cases} 3x-4y=8 \\ 5x+4y=-24 \end{cases}$

__**Answer**__:

If we try to solve this system by substitution, we realize that there is no variable with a coefficient of 1 or -1. Therefore if we try to solve one of the equations for x or y, we will be working with fractions, which can be messy. It turns out that this system is perfectly set up for solving by the second algebraic method: elimination.

Notice that the y term in each equation is exactly the same with opposite signs. If we add the two equations, we will eliminate the y terms and the result will be one equation with x as the only variable. Thus, we have **eliminated** one of the variables. We can add both equations and solve the resulting equation because of the addition property of equations – we are adding equal expressions to both sides.

$$
\begin{array}{ll}
3x-4y=8 & \\
\underline{5x+4y=-24} & \text{Add the equations vertically} \\
8x=-16 & \text{divide both sides by 8} \\
x=-2 &
\end{array}
$$

The result is $x=-2$ but that is not the final solution. We now need to find the value of y. We select one of the original equations and replace x with –2 resulting in:

$$
\begin{array}{ll}
3(-2)-4y=8 & \\
-6-4y=8 & \text{add 6 to both sides} \\
-4y=14 & \\
y=\dfrac{-14}{4}=\dfrac{-7}{2}=-3.5 &
\end{array}
$$

The solution is: $(-2,\ -3.5)$.

Remember to **check** that this ordered pair makes each original equation true.

Practice Exercise: Solving systems of equations by elimination

For exercises 1 – 2, solve each system of equations by elimination and check each solution.

1) $\begin{cases} 4x-3y=-7 \\ 5x+3y=25 \end{cases}$ **2)** $\begin{cases} 2x-3y=-7 \\ -2x-3y=25 \end{cases}$

Solving by Elimination if Opposite Coefficients are not Given

In the previous examples one variable had opposite coefficients. Adding the two equations together eliminated that variable completely. This allowed us to solve for the other variable. This is the idea behind the elimination method. However, generally we won't have opposites in front of one of the variables. In this case we will manipulate the equations to get the opposites we want by multiplying one or both equations (on both sides!) This is shown in the next set of examples.

Example 2:

Solve the following system of equations by elimination: $\begin{cases} 5x-y=4 \\ 4x+2y=-1 \end{cases}$

<u>**Answer**</u>:

If we just add the two equations, we will not eliminate one variable. However, if the first equation had – $2y$ instead of $-y$, we would have the y term in each equation with opposite coefficients.

In the first equation, if we **multiply every term on both sides** by 2, then we will have – $2y$.

$$\begin{array}{lcl} 2(5x-y=4) & \rightarrow & 10x-2y=8 \\ \underline{4x+2y=-1} & & \underline{4x+2y=-1} \\ & & 14x=7 \\ & & x=\frac{7}{14}=\frac{1}{2} \end{array}$$

Now we can add the two equations and eliminate y.

The result is $x=\frac{1}{2}$ but that is not the final solution. We now need to find the value of y. We select one of the original equations and replace x with $\frac{1}{2}$ resulting in:

$$4\left(\frac{1}{2}\right)+2y=-1$$

$2+2y=-1$ Subtract 2 from both sides

$$2y=-3$$

$$y=\frac{-3}{2}$$

The solution is: $\left(\frac{1}{2},\frac{-3}{2}\right)$.

Remember to **check** that this ordered pair makes each original equation true.

Example 3:

Solve the following system of equations by elimination: $\begin{cases} -6x+5y=22 \\ 2x+3y=2 \end{cases}$

Answer:
If we just add the two equations, we will not eliminate one variable. However, if the second equation had $6x$ instead of $2x$, we would have the x term in each equation with opposite coefficients.
In the second equation, if we **multiply every term on both sides** by 3, then we will have $6x$.

$$\begin{array}{lcl} -6x+5y=22 & & -6x+5y=22 \\ 3\underline{(2x+3y=2)} & \rightarrow & \underline{6x+9y=6} \\ & & 14y=28 \\ & & y=2 \end{array}$$

Now we can add the two equations and eliminate x.

The result is $y=2$ but that is not the final solution. We now need to find the value of x. We select one of the original equations and replace y with 2 resulting in:

$2x+3(2)=2$

$2x+6=2$

$2x=-4$

$x=-2$

The solution is: (– 2, 2).
Remember to **check** that this ordered pair makes each original equation true.

Example 4:

Solve the following system of equations by elimination: $\begin{cases} 6x-5y=14 \\ 2x+2y=-10 \end{cases}$

Answer:
If we just add the two equations, we will not eliminate one variable. However, if the second equation had $-6x$ instead of $2x$, we would have the x term in each equation with opposite coefficients. We will multiply every term of the second equation (both sides) by -3.

$$\begin{array}{lcl} 6x-5y=14 & & 6x-5y=14 \\ -3\underline{(2x+2y=-10)} & \rightarrow & \underline{-6x-6y=30} \\ & & -11y=44 \\ & & y=-4 \end{array}$$

Now we can add the two equations and eliminate x.

Divide by -11

The result is $y = -4$ but that is not the final solution. We now need to find the value of x. We select one of the original equations and replace y with – 4 resulting in:

$2x+2(-4)=-10;\quad 2x-8=-10;\quad 2x=-2;\quad x=-1$

The solution is: (– 1, – 4). Remember to **check** this solution in both equations.

Practice Exercise: Solving systems of equations by elimination

For exercises 3 – 6, solve each system of equations by elimination and check each solution.

3) $\begin{cases} 3x + y = 1 \\ 5x + y = 3 \end{cases}$

4) $\begin{cases} x + 4y = 1 \\ x - 2y = -5 \end{cases}$

5) $\begin{cases} 2x + 3y = -10 \\ -x + 2y = -2 \end{cases}$

6) $\begin{cases} 5x - 3y = 1 \\ 8x - 6y = 4 \end{cases}$

Solving by Elimination if Both Equations need to be Multiplied by a Constant

When we are not given a variable with opposite coefficients, what we are looking for is the least common multiple (LCM) of the coefficients. We can eliminate either variable, as long as we have opposite coefficients. This illustrates an important point: in some problems we will have to multiply both equations by a constant (on both sides) to get the opposites we want.

Example 5:

Solve the following system of equations by elimination: $\begin{cases} 3x + 6y = -9 \\ 2x + 9y = -26 \end{cases}$

__**Answer**__:
If we just add the two equations, we will not eliminate one variable. Notice that we will have to multiply both equations by a constant. We can eliminate either variable. To eliminate x, we will need to get the LCM of 3 and 2, which is 6. To get opposite coefficients, we will multiply the first equation by 2 and the second equation by -3.

$$\begin{array}{lcl} 2(3x+6y=-9) & \rightarrow & 6x+12y=-18 \\ \underline{-3(2x+9y=-26)} & \rightarrow & \underline{-6x-27y=78} \end{array}$$ Now we can add the two equations and eliminate x.

$$-15y = 60$$ Divide by -15

$$y = -4$$

The result is $y = -4$ but that is not the final solution. We now need to find the value of x. We select one of the original equations and replace y with -4 resulting in:
$3x + 6(-4) = -9;\;\; 3x - 24 = -9;\;\; 3x = 15;\;\; x = 5$

The solution is: (5, – 4). Remember to **check** this solution in both equations.

Once we decide to solve a system by elimination, we need to make sure that all variables and constants are lined up before we start multiplying and adding equations. Both equations should be in general form ($Ax + By = C$). This is illustrated in the next example.

Example 6:

Solve the following system of equations by elimination: $\begin{cases} 2x-5y=-13 \\ -3y+4=-5x \end{cases}$

__Answer__:

We need to write both equations so that the like variable terms line up vertically in a column. The first equation is written in general form. For the second equation, we will add $5x$ to both sides so that the x and y terms are on the same side. Then we will subtract 4 from both sides so that the constant is alone on the right side of that equation. The result is:

$$\begin{cases} 2x-5y=-13 \\ 5x-3y=-4 \end{cases}$$

We need to get opposite coefficients for either x or y. Let's eliminate x by introducing the LCM for 2 and 5, which is 10.

$$5(2x-5y=-13) \quad \rightarrow \quad 10x-25y=-65$$

$$\underline{-2(5x-3y=-4)} \quad \rightarrow \quad \underline{-10x+6y=8} \qquad \text{Now we can add the two equations and eliminate } x.$$

$$-19y=-57 \qquad \text{Divide by } -19$$

$$y=3$$

The result is $y = 3$ but that is not the final solution. We now need to find the value of x. We select one of the original equations and replace y with 3 resulting in:

$2x-5(3)=-13; \quad 2x-15=-13; \quad 2x=2; \quad x=1$

The solution is (1, 3). Remember to **check** this solution in both original equations.

Practice Exercises: Solving by elimination if both equations need to be multiplied by a constant.

For exercises 7 – 10, solve each system of equations by elimination and check each solution.

7) $\begin{cases} 3x-5y=9 \\ 4x+8y=12 \end{cases}$

8) $\begin{cases} 5x-2y=-13 \\ 2x-3y=-14 \end{cases}$

9) $\begin{cases} 3x-6y=0 \\ 5x-4y=-6 \end{cases}$

10) $\begin{cases} 2x-5y=19 \\ 5x+3y=1 \end{cases}$

Steps for Solving a System of Linear Equations by Elimination

We can summarize the process for solving a system of linear equations by elimination:

1) **Make sure to write the equations in general form, so that the like terms line up vertically.**
2) **Multiply one or both equations to create opposite coefficients for one variable.**
3) **Add the resulting equations by combining like terms. This should eliminate one variable. Solve for the remaining variable.**
4) **Substitute the solution from step 3 into either of the two original equations, and solve for the other variable. State the solution as an ordered pair (x, y) or in "$x =$ ___, $y =$ ___" form.**
5) **Check the solution in both of the original equations.**

Solving Systems with Infinite Solutions or No Solution

Just as with graphing and substitution, it is possible to have no solution or infinite solutions with elimination. In the same way as in substitution, if the variables all disappear from our problem, a true statement will indicate infinite solutions and a false statement will indicate no solutions.

Example 7:

Solve the following system of equations by elimination: $\begin{cases} 2x-5y=3 \\ -6x+15y=-9 \end{cases}$

Answer:
To get opposite coefficients, it will be easiest to multiply the first equation by 3.

$$\begin{array}{lcr} 3(2x-5y=3) & \rightarrow & 6x-15y=9 \\ \underline{-6x+15y=-9} & \rightarrow & \underline{-6x+15y=-9} \\ & & 0=0 \end{array}$$

Now we can add the two equations and eliminate x.

True! Therefore there are infinite solutions to this system. These two lines are coincident.

Example 8:

Solve the following system of equations by elimination: $\begin{cases} 4x-6y=8 \\ 6x-9y=15 \end{cases}$

Answer:
The LCM for x's is 12, so we will multiply the first equation by 3 and the second equation by – 2.

$$\begin{array}{lcr} 3(4x-6y=8) & \rightarrow & 12x-18y=24 \\ \underline{-2(6x-9y=15)} & \rightarrow & \underline{-12x+18y=-30} \\ & & 0=-6 \end{array}$$

Now we can add the two equations and eliminate x.

False! Therefore there are no solutions to this system. These two lines are parallel.

Deciding Which Method to use when Solving a Linear System of Equations

We have covered three different methods that can be used to solve a system of two equations with two variables: graphically or algebraically, by substitution or elimination. While all three methods can be used to solve any system, graphing works best for small integer solutions, or when the directions specify that the system must be solved by graphing.

The algebraic methods are substitution and elimination. Substitution works well when one of the equations is written as x = or y =, or when one of the variables in an equation has a coefficient of 1 or -1. Elimination works well when the equations are written in general form: $Ax + By = C$. Because each method has its own strengths, it is important you are familiar with all three methods.

Practice Exercise: Solving systems of equations by elimination

For exercises 11 – 20, solve each system of equations by elimination and check each solution.

11) $\begin{cases} x+6y=8 \\ -x-2y=0 \end{cases}$

12) $\begin{cases} 6x-5y=14 \\ 2x+2y=-10 \end{cases}$

13) $\begin{cases} 4x+8y=8 \\ 3x+6y=6 \end{cases}$

14) $\begin{cases} 2x+y=-1 \\ 3x-y=-9 \end{cases}$

15) $\begin{cases} 3x+4y=25 \\ 3x-3y=4 \end{cases}$

16) $\begin{cases} 2x-y=1 \\ 4x-2y=4 \end{cases}$

17) $\begin{cases} x-2y=-2 \\ -2x+4y=4 \end{cases}$

18) $\begin{cases} 5x+2y=8 \\ 3x-5y=11 \end{cases}$

19) $\begin{cases} 4x+3y=1 \\ 5x-4y=9 \end{cases}$

20) $\begin{cases} 3x+2y=-4 \\ 4x=5y+10 \end{cases}$

Using a System of Equations to Solve Applications

When solving practical problems, it is often more convenient to introduce two variables rather than only one. Two variables should be introduced only when two relationships can be found within the problem. Each relationship will produce an equation, and a system of two equations in two variables will result.

The following is a five-step method for solving linear applications with two variables.

1) **Introduce two variables, one for each unknown quantity.**
2) **Look for two relationships within the problem. Translate the verbal phrases into mathematical expressions to form two equations.**
3) **Solve the resulting system of equations.**
4) **Check the solution.**
5) **Write a conclusion.**

Example 9:

Tickets for a college's magic show were being sold for a student price of \$2.50. Tickets for non-students cost \$3.50. The total amount collected was \$537. If 200 tickets were sold, how many student and how many non-student tickets were sold?

Answer:

1) The question: "how many student and how many non-student tickets were sold?" helps us define the variables. Therefore the variables can be defined as: let x = the number of student tickets sold and y = the number of non-student tickets sold.

2) From the problem, the number of student tickets plus the number of non-student tickets is 200 tickets, so an equation that represents tickets is: $x + y = 200$. A second equation comes from the amount collected. Money from student tickets (2.50 x) plus money from non-student tickets (3.50 y) is the total amount collected, \$537. So the second equation is: $2.5x + 3.5y = 537$.

3) To solve the system, $\begin{cases} x + y = 200 \\ 2.5x + 3.5y = 537 \end{cases}$ we will multiply the first equation by –2.5 to eliminate x when the two equations are added.

$$\begin{array}{lcr} -2.5(x + y = 200) & \rightarrow & -2.5x - 2.5y = -500 \\ \underline{2.5x + 3.5y = 537} & & \underline{2.5x + 3.5y = 537} \\ & & y = 37 \end{array}$$

Substitute $y = 37$ into the first equation, and the result is $x = 163$.

4) Checking the solution, $x = 163$ and $y = 37$, we find that this solution makes each of the original equations true:
$$\begin{aligned} (163) + (37) &= 200 \\ 2.5(163) + 3.5(37) &= 537 \end{aligned}$$

5) Therefore, since x represents the number of student tickets sold and y represents the number of non-student tickets sold, we can state that 163 student tickets were sold and 37 non-student tickets were sold.

Example 10:

A parking meter contains 42 coins. The total value of the coins is \$8.40. If the meter contains only dimes and quarters, how many of each type of coin are there in the parking meter?

<u>**Answer**</u>:

1) The question: "how many of each type of coin" asks for the number of dimes and the number of quarters that are in the parking meter. Therefore the variables can be defined as: let x = the number of dimes and y = the number of quarters.

2) From the problem, the number of dimes plus the number of quarters is 42 coins, so an equation that represents the number of coins is: $x+y=42$. A second equation comes from the amount of money in the parking meter. Since the value of a dime is \$0.10 and the value of a quarter is \$0.25, the money from dimes ($0.10x$) plus money from quarters ($0.25y$) is the total amount collected, \$8.40. So the second equation is: $0.10x+0.25y=8.40$.

3) To solve the system, $\begin{cases} x+y=42 \\ 0.10x+0.25y=8.40 \end{cases}$ we will multiply the first equation by -0.10 to eliminate x when the two equations are added.

$$\begin{array}{rcr} -0.10(x+y=42) & \rightarrow & -0.10x-0.10y=-4.2 \\ \underline{0.10x+0.25y=8.40} & & \underline{0.10x+0.25y=8.40} \\ & & 0.15y=4.2 \\ & & y=28 \end{array}$$

Substitute $y = 28$ into the first equation, and the result is $x = 14$.

4) Checking the solution, $x = 14$ and $y = 28$, we find that this solution makes each of the original equations true:

$$(14)+(28)=42$$
$$0.10(14)+0.25(28)=8.40$$

5) Therefore, since x represented the number of dimes and y represented the number of quarters, we can state that there were 14 dimes and 28 quarters in the parking meter.

3.5 SUMMARY

- **To solve a system of linear equations by elimination, both equations should be in the** $Ax + By = C$ **form. The goal is to add the two equations and eliminate one of the variables. For this to happen, the coefficients of one of the variables must be opposite.**

Example: $\begin{cases} 2x - 3y = 7 \\ -6x + 2y = 21 \end{cases}$

WORK	STEP
	1. Make sure to write the equations so that the like terms line up vertically.
	2. Multiply one or both equations to create opposite coefficients on one variable.
	3. Add the resulting equations by combining like terms. This should eliminate one variable. Solve for the remaining variable.
	4. Substitute the solution from step 3 into either of the two original equations, and solve for the other variable. State the solution as an ordered pair (x, y) or in "$x =$ ___, $y =$ ___" form.
	5. Check the solution in both of the original equations.

Interpreting Results From the Elimination Method

- There is ***one solution*** (x, y) to the system when a value can determined for both x and y.
- There are ***infinite solutions*** to the system when all the variables add to zero and a true statement remains (for example: $0 = 0$).
- There are ***no solutions*** to the system when all the variables add to zero and a false statement remains (for example: $0 = 5$).

Example: Solve the following system of equations by elimination: $\begin{cases} 5x - 15y = 45 \\ -x + 3y = -9 \end{cases}$

Applications:
When solving an application by using a system of equations, we will often use a diagram to help understand the problem. Once we define our two variables, we may need to use a formula. Finally, we will translate key words into an equation.

Example:
The perimeter of a rectangle is 76 inches. Its length is two inches less than three times its width. What are the dimensions of the rectangle?

In this case, we can draw a diagram to illustrate:

length is 2 less than 3 times the width

Perimeter is 76 inches

width

Work	Step
	1. Define variables to represent the length and width. Write an equation expressing the perimeter of the rectangle in terms of the length and the width.
	2. Write the system of equations expressing the length and the width.
	3. Solve this system of linear equations from step 2 by substitution. Solve for the width.
	4. Substitute the value into either of the original equations. Solve for the length.
	5. State your solution clearly.
	6. Check the solution in your two equations.

Applications:

Example:

At a back-to-school sale, Lydia bought 3 pens and 5 notebooks and paid $11.42 before tax. Carolina bought 4 of the same pens and 2 of the same notebooks and paid $5.94 before tax. How much did each pen cost? How much did each notebook cost?

In this example, a system of equations is appropriate. First, define the two variables.

Let x = ____________________________________

Let y = ____________________________________

Each of the first two sentences contains three numbers.

"…Lydia bought 3 pens and 5 notebooks and paid $11.42…"

"Carolina bought 4 of the same pens and 2 of the same notebooks and paid $5.94…"

Those three numbers become A, B, and C in our standard $Ax + By = C$ equations.
Thus, our system of equations is:

$$\begin{cases} \\ \\ \end{cases}$$

Solve this system of equations by elimination.

<u>Answer</u>:
Each pen costs __________.
Each notebook costs __________.

Check:

3.6 Statistics

Statistics is an important part of everyday life, whether you are starting a new business, deciding how to plan for your financial future, or watching the news on television. Statistics comes up in everything from opinion polls, to economic reports, to the research on cancer prevention. Understanding the basic ideas behind statistics is crucial to your success in the modern world.

Types of Data

As we study descriptive statistics, you will learn about two types of data:

- ***Qualitative data*** – consists of attributes, labels, or non-numerical entries. Examples of qualitative data are gender, eye color, ethnic group, or telephone number.
- ***Quantitative data*** – consists of numerical measurements or counts. Examples of quantitative data are age, height, weight, or number of classes taken.

Organizing Data

When given a large set of data, it is important to be able to organize it. As we organize data, the following definitions will be helpful:

- $\sum$ ***(sigma)*** – The uppercase Greek letter which indicates a summation of values.
- The ***frequency*** (f) of a class is the number of data entries in the *class*. $\sum f = n$.
- The ***relative frequency*** (rf) of a class is the portion or percentage of the data that falls in that class. To find the relative frequency of a class, divide the frequency f by the sample size n. $rf = \frac{f}{n}$.
- ***Bar graph*** – a graph to display qualitative data.
- ***Histogram*** – a graph to display quantitative data.

We illustrate each of these concepts in the following example.

Example 1:

A MATH 131 statistics student is preparing a presentation for the Student Achievement Showcase at Brookdale. She is planning to analyze the relationship between the sugar and fat content in different brands of cereals.

The data set she collected from a supermarket follows. Fat and sugar content is given in grams (g).

	Manufacturer	Fat (g)	Sugar (g)
1.	Nabisco	1	6
2.	Kellogg's	1	5
3.	Kellogg's	0	0
4.	Post	0	5
5.	General Mills	2	1
6.	General Mills	0	3
7.	General Mills	0	2
8.	Nabisco	0	0
9.	Kellogg's	0	3
10.	Post	1	5
11.	Post	0	3
12.	Post	3	4
13.	Kellogg's	1	6
14.	General Mills	1	3
15.	General Mills	1	6

Use the cereal data set to answer the following questions:

a) Name the **qualitative** variable.
b) Prepare a frequency distribution table for the **qualitative** data.
c) Prepare a Bar graph.
d) Name the **quantitative** variables.
e) Prepare a frequency (f) and relative frequency (rf) distribution table for the quantitative variable: sugar.
f) Use the table in part e) to prepare a frequency histogram or a relative frequency histogram.

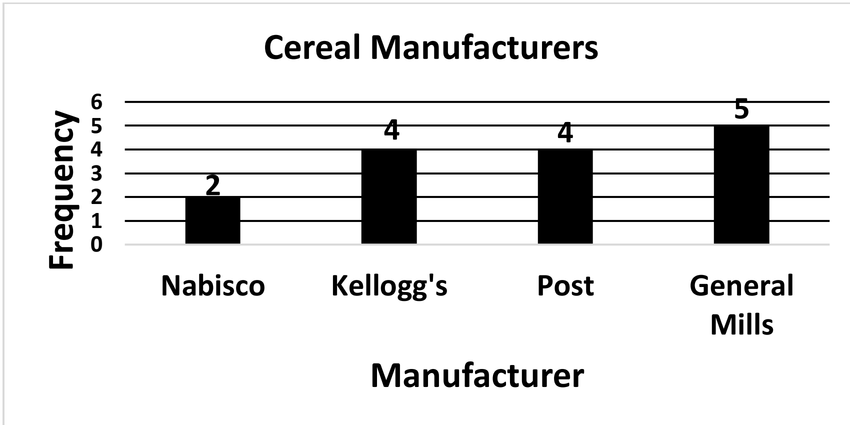
Answers:

a) Manufacturer
b)

Qualitative Variable: Manufacturer	**Tally**	**Frequency (f)**
Nabisco	//	2
Kellogg's	////	4
Post	////	4
General Mills	~~////~~	5

$$\sum f = 15$$

c)

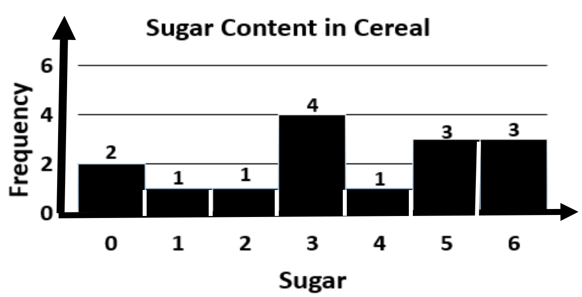


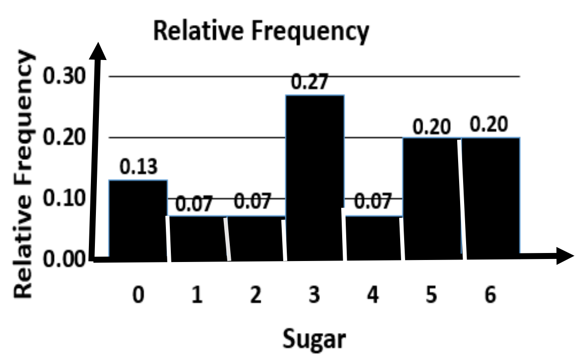
d) fat, sugar

e)

Quantitative variable: sugar	Tally	Frequency (f)	Relative frequency (rf)
0	//	2	$2/15 \approx 0.13$
1	/	1	$1/15 \approx 0.07$
2	/	1	$1/15 \approx 0.07$
3	////	4	$4/15 \approx 0.27$
4	/	1	$1/15 \approx 0.07$
5	///	3	$3/15 = 0.2$
6	///	3	$3/15 = 0.2$
		$\sum f = 15$	$\sum rf = 1$

f)





Mean, Median, and Mode

If, for example, you are finding the average grade for 5 quizzes (by adding the 5 grades and then dividing that sum by 5), you are finding the mean for that set of data. But what if your quiz grades were: 100, 100, 100, 100 and 10. Does an average of 82 reflect what you know on all five quizzes? It is helpful to have other ways to find an average of data, so that we can better understand the data set.

The following definitions are important:

- The ***mean*** of a data set is the sum of the data entries divided by the number of entries.
    - ❖ To find the ***mean*** of a data set, use the formula, $\bar{x} = \dfrac{\sum x}{n}$.
- The ***median*** of a data set is the value that lies in the middle of the data set when the data set is ordered.
    - ❖ To find the ***median***:
        - o Rearrange data from low to high
        - o If the data set has an odd number of entries, the median is the middle data entry.
        - o If the data set has an even number of entries, the median is the mean of the two middle data entries.
- The ***mode*** of a data set is the data entry that occurs with the greatest frequency.
    - ❖ To find the ***mode***, find the data entry that occurs with the greatest frequency.

Example 2:
Use the cereal data from Example 1 to answer the following questions.

a) Find the mean of the quantitative variable: sugar.

b) Find the median of the quantitative variable: sugar.

c) Find the mode of the quantitative variable: sugar.

d) Find the mean, median and mode for the other **quantitative** variable: fat.

Answers:

a) To find $\sum x$ which is the total amount of grams of sugar in the cereals, you can add the numbers in the Sugar column. An easier way may be to use the Frequency Table found in Example 1e). There were **2** cereals that had 0 grams of sugar, **1** cereal had 1 g sugar, **1** cereal had 2 g sugar, **4** cereals had 3 g sugar, **1** cereal had 4 g sugar, **3** cereals had 5 g sugar, and **3** cereals had 6 g sugar. To find the total amount of grams of sugar, multiply the frequency by the number of grams. So we have:

$$\sum x = 2(0)+1(1)+1(2)+4(3)+1(4)+3(5)+3(6)=52$$

The mean is $\bar{x} = \dfrac{\sum x}{n} = \dfrac{52}{15} = 3.47$. Therefore the average amount of sugar in the cereal selected for the data set is 3.47 g.

b) Writing the grams of sugar content in order from least to greatest, we have: 0, 0, 1, 2, 3, 3, 3, 3, 4, 5, 5, 5, 6, 6, 6. For the 15 boxes of cereal in this data set, the middle sugar will be. $\frac{15}{2} = 7.5 = 8^{th}$ place. The eighth number of the list of grams of sugar is: 3. Therefore the median is 3.

c) The number of grams of sugar that occurs the most is 3 grams. So the mode is 3.

d) For fat: $\sum x = 7(0) + 6(1) + 1(2) + 1(3) = 11$

The mean is $\bar{x} = \frac{\sum x}{n} = \frac{11}{15} = 0.73$. Therefore the average amount of grams of fat in the cereal in this data set is 0.73 g.

The median is found by placing the fat content in order from least to greatest: 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 2, 3. The eighth number in this list is 1. Therefore the median is 1.

The number of grams of fat that occurs the most is 0 g. Therefore the mode is 0.

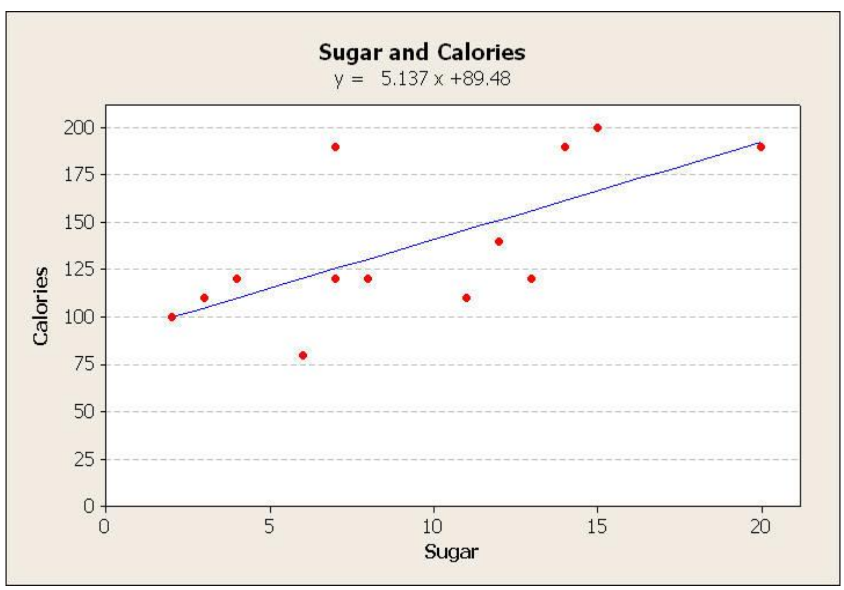
Regression Line

- A regression line $y = mx + b$ can be used to predict the value of y for a given value of x.

Example 3:

The following table and scatter plot give information on the amount of sugar (in grams) and the calorie count in one serving of a sample of 13 varieties of Kellogg's cereal.

Sugar, x	*Calories, y*
4	120
15	200
12	140
11	110
8	120
6	80
7	190
2	100
7	120
14	190
20	190
3	110
13	120



a) The regression line for calories and sugar for the table above is $y = 5.137x + 89.48$. What is the slope? Give an interpretation of the slope.

b) What is the y-intercept? Give the meaning of the y-intercept in the context of this regression line.

c) What is the predicted number of calories for a cereal that has 5 grams of sugar per serving?

d) What is the predicted number of calories for a cereal that has 16 grams of sugar per serving?

Answers:

a) $m = 5.137$. There is an increase of 5.137 calories for every gram of sugar.

b) Since $b = 89.48$, the y-intercept is (0, 89.48). We can predict that if there are 0g sugar in a cereal serving, then it has 89.48 calories.

c) $y = 5.137(5) + 89.48 = 115.165$ Therefore we can predict that a cereal with 5g sugar per serving is 115.165 calories.

d) $y = 5.137(16) + 89.48 = 171.672$ Therefore we can predict that a cereal with 16g sugar per serving is 171.672 calories.

Practice Exercise: Statistics

1) A professor asked the twenty students in her Statistics class to write down the average number of hours they sleep each night, to the nearest half hour. The results are in the table below.

Student	Hours of Sleep
Lisa	4
Carolina	7.5
Lydia	5
Raemin	7.5
Mian	6
Jay	6.5
Bobby	6
Clay	8
Jade	4.5
Reo	5.5

Student	Hours of Sleep
Mariama	6
Deslyn	5
Celina	5.5
Daniella	8
Stephanie	7.5
Elise	7.5
Paolo	6
Emma	8
Brooke	7.5
Angelica	7.5

a) Name the quantitative variable.____

b) Create a frequency distribution table and histogram for the hours of sleep.

c) Find the mean, median and mode for the hours of sleep of the Statistics students.

d) What is the relative frequency for 6 hours of sleep?

2) A random sample of the income of ten professional athletes produced a set of data, where x is the number of commercial endorsements the player has and y is the amount of money made (in millions of dollars). The regression line that relates this data is: $y = 2.23 + 1.00x$.

a) What is the slope of this line? Give an interpretation of the slope.

b) What is the y-intercept? Explain what the y-intercept means for this data.

c) If a professional athlete has 7 endorsements, how much money will she be expected to make?

d) How many endorsements should a professional athlete have if she wants to have an income of approximately 5 million dollars?

3.6 SUMMARY

1. Classify the following data as qualitative or quantitative.

a) Name ____________________

b) Age ____________________

c) Height ____________________

d) Eye Color ____________________

e) Phone number ____________________

2. A __________________ is a graph to display qualitative variables.

3. A __________________ is a graph to display quantitative variables.

4. State True/False for the following statements.

a) Mean of a data set always exists.

b) Mode of a data set always exists.

c) An outlier (a really high or really low data value) has no effect on mean of the data.

d) To find median, the data must be arranged in an increasing or decreasing order.

5. For an auto insurance company the premium is given by the following equation: $y = 250 - 10x$ *where* x is the number of years of accident-free driving and y is the quarterly premium.

a) What is the slope for this equation? Interpret the slope for this situation.

b) What is the y-intercept for this equation? Interpret the y-intercept for this situation.

3.7 Probability

Probability is used every day in the news, commercial advertising, to present data and even to play games. You may already have an idea of what probability means. In this section we will learn how to calculate probability in a systematic manner, as well as how to calculate some simple probabilities.

Think about the type of answer you expect to hear, when you are given questions such as: "What is the probability that the Mets will win the World Series?" or "What is the probability your phone will ring today?" or "What is the probability that it will rain today?" or "What is the probability that the temperature will reach 130 degrees today?"

Usually, the answer to these questions is given as a percent. For example:

- The Mets have a 90% chance of winning the World Series.
- The probability of my phone ringing today is 100%.
- The probability that it will rain today is 20%.
- The probability that the temperature will reach 130 degrees today is 0%.

Each of these probabilities represent a number that is between 0 and 1. Probability is a branch of mathematics that attempts to **quantify** uncertainty.

As we study probability, the following definitions will be useful:

- **Probability experiment** is any procedure that can be infinitely repeated and has a well-defined set of possible outcomes, known as the sample space.
- **Outcome:** The result of a single trial in a probability experiment is called an outcome. For example, if the experiment is to flip a coin, one of the outcomes is Head.
- **Sample Space:** The set of all possible outcomes of an experiment. For example, when you flip a coin, the sample space is {Head, Tail}.
- **Tree Diagram:** a diagram that shows all the possible outcomes of an event. Introduce tree diagrams to list sample spaces for simple experiments like flipping a coin twice or rolling a die followed by flipping a coin.
- **Event:** A subset of sample space is called an event. It may consist of one or more outcomes.

Sample Space and Tree Diagrams

Example 1:

List the sample space for rolling a single six-sided die. How many outcomes are there in this sample space?

<u>Answer</u>:

S = {1, 2, 3, 4, 5, 6}   There are 6 outcomes in this sample space.

Example 2:

List the sample space if you flip a coin twice. How many outcomes are there in this sample space?

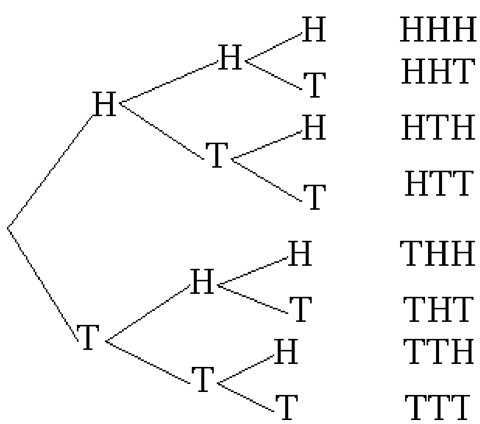
<u>Answer</u>:

S = {HH, HT, TH, TT} There are 4 outcomes in this sample space.

Example 3:

Use a tree diagram to list the sample space if you flip a coin three times.

<u>Answer</u>:



S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}
There are 8 outcomes in this sample space.

The Counting Principle

The Fundamental Counting Principle: If there are m ways to do one step of an experiment and n ways to do the second step, there are $m \cdot n$ ways to do both steps.

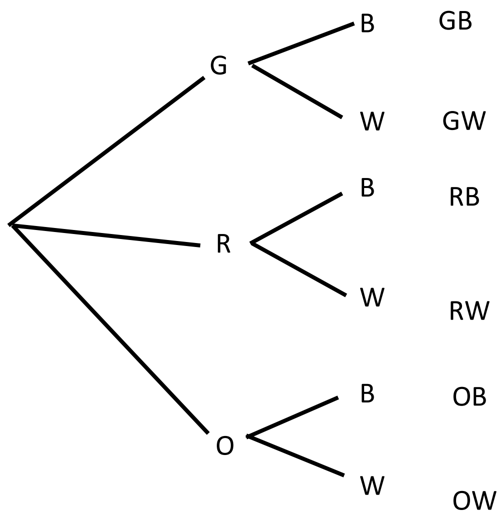
Example 4:

Alberto needs to wear a uniform to his job. He has 3 different uniform shirts (green, red and orange) and 2 different uniform pants (black and white) with the logo of the company.

- **a)** Use a tree diagram to list all possible outfits he can make.
- **b)** List the sample space.
- **c)** Use the Fundamental Counting Principle to find the number of different outfits Alberto can wear to his job.

Answers:

a)



b) Sample space: {GB, GW, RB, RW, OB, OW}

c) 3(2) = 6. Alberto can wear 6 different outfits.

Example 5:

Barbara has 6 pants and 5 shirts that she can wear to work. How many different work outfits are possible for Barbara?

Answer:

6(5) = 30 Barbara has 30 different work outfits.

Example 6:

Ju Wen is working on a project where she needs to create garage door codes. Each code is required to have one letter from the alphabet (A – Z) followed by three digits from 0 through 9. Repeating a number is acceptable. How many different codes can Ju Wen generate?

Answer:

Since there are 26 letters in the alphabet and 10 possible numbers, the number of different codes is:

26(10)(10)(10) = 26,000 Therefore Ju Wen can generate 26,000 different codes.

Example 7:

The Brookdale Cafeteria is offering a "lunch deal" for $6 during the summer. Each lunch deal offers a main entrée, a drink, a soup and a dessert. If there are 5 main entrées, 7 drinks, 3 soups and 2 desserts to choose from, how many different lunch deal options are there?

<u>Answer</u>:

$5(7)(3)(2) = 210$ Therefore there are 210 lunch deal options.

Practice Exercises: Tree diagrams and the counting principle

For exercises 1 – 2, complete a tree diagram that lists the outcomes in a sample space. Then use the Fundamental Counting Principle to evaluate (and check) your answer.

1) Reo wants to know how many outcomes there are if he tosses a coin and then tosses a die.
  - **a)** Use a tree diagram to list the sample space for tossing a coin and then tossing a die.
  - **b)** How many different outcomes are there to Reo's experiment?
  - **c)** Use the Fundamental Counting Principle to find how many outcomes there are to an experiment that tosses a coin and then tosses a die.

2) Baby Carolina has 5 terry sleep and play onesies in pink, blue, yellow, green and lavender. She also has 2 vests, in orange and in magenta. If an outfit consists of a onesie and a vest,
  - **a)** use a tree diagram to list the sample space for the different outfits for Carolina.
  - **b)** How many different outfits are there for Carolina?
  - **c)** Use the Fundamental Counting Principle to find how many outfits (consisting of a onesie and vest), there are for Carolina.

There are two kinds of probabilities: **Theoretical Probability** and **Empirical Probability**.

Theoretical Probability

Theoretical probability is the probability that we expect theoretically. For example, if we toss a coin, in theory we would expect that the probability of getting a head is 50%. In other words, when we toss a coin, we can expect that $\frac{1}{2}$ of the outcomes will be heads.

The probability of an event E for a sample space with equally likely outcomes is given by:

$$P(E) = \frac{\text{number of favorable outcomes in E}}{\text{total number of outcomes in the sample space}}$$

Example 8:

Find the probability of an even number when you roll a six-sided die once.

Answer:

Since there are 3 even numbers on a six-sided die, the probability of an even number is $\frac{3}{6}$ or $\frac{1}{2}$ or 0.5.

Example 9:

Find the probability of rolling a number greater than 5 when you roll a six-sided die once.

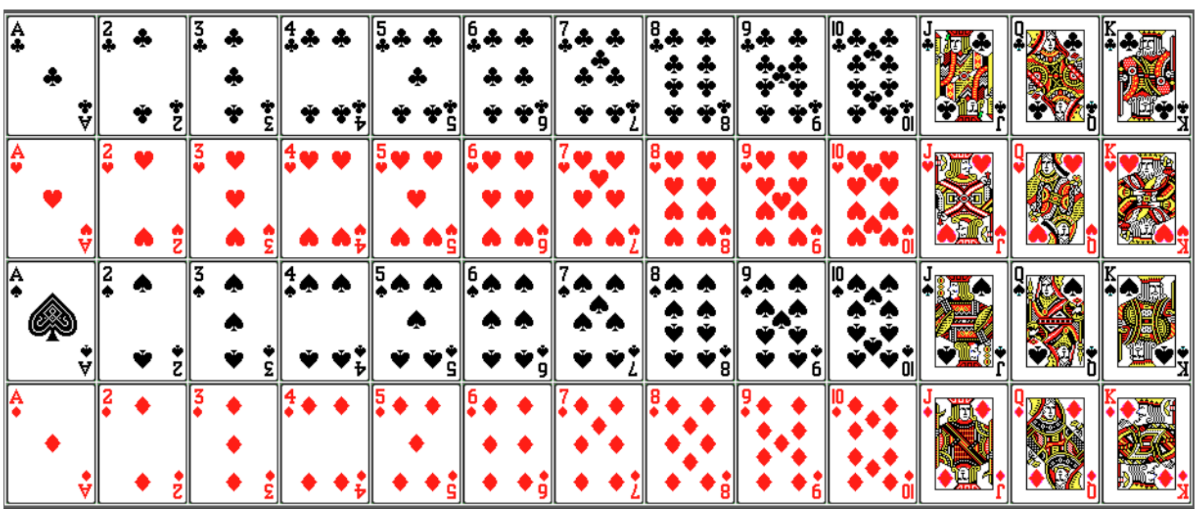
Answer:

There is only one number greater than 5 on a six-sided die, so the probability is $\frac{1}{6}$ or 0.1667.

Example 10:

Note: For this example, you need to know the contents of a deck of cards. For a quick reference, an image of all cards in a deck is included.

Quick facts: There are 52 cards in a deck and 4 suits: clubs, hearts, spades and diamonds.



If a card is drawn randomly from the deck, find the probability that:

a) it is a king.
b) it is a spade.
c) it is the queen of hearts.
d) it is not a diamond.

Answers:

a) There are 4 kings in a deck, so the probability that the card is a king is $\frac{4}{52}$ or $\frac{1}{13}$ or 0.077.

b) There are 13 spades, so the probability that the card is a spade is $\frac{13}{52}$ or $\frac{1}{4}$ or 0.25.

c) There is only one queen of hearts in the deck, so the probability of picking a queen of hearts is $\frac{1}{52}$ or 0.019

d) There are 13 diamonds in the deck. The number of cards that are not a diamond is 52 – 13 which is 39. So the probability that the card is not a diamond is $\frac{39}{52}$ or $\frac{3}{4}$ or 0.75.

Empirical Probability

Empirical probability is the probability that we actually observe when an experiment is carried out. The probability of an event *E* for a sample space with equally likely outcomes is given by:

$$P(E) = \frac{\text{number of times a favorable outcome is observed in an experiment}}{\text{total number of trials}}$$

Example 11:

The following is the data set collected from a supermarket that you saw in the previous class. Use this table to answer the following questions.

	Manufacturer	Fat	Sugars
1.	Nabisco	1	6
2.	Kellogg's	1	5
3.	Kellogg's	0	0
4.	Post	0	5
5.	General Mills	2	1
6.	General Mills	0	3
7.	General Mills	0	2
8.	Nabisco	0	0
9.	Kellogg's	0	3
10.	Post	1	5
11.	Post	0	3
12.	Post	3	4
13.	Kellogg's	1	6
14.	General Mills	1	3
15.	General Mills	1	6

If a random cereal box is picked from the 15 cereals listed here, what is the probability that:

a) it is manufactured by General Mills?
b) it has no sugar?
c) it has 1 gram of fat per serving?
d) it has no more than 3 grams of sugar?

Answers:

a) Because 5 cereals were manufactured by General Mills, the probability that the cereal is from General Mills is $\frac{5}{15}$ or $\frac{1}{3}$ or 0.33.

b) Two cereals had no sugar, so the probability of selecting a cereal with no sugar is $\frac{2}{15}$ or 0.13.

c) Six cereals have 1 gram of fat per serving, so the probability of selecting that type of cereal is $\frac{6}{15}$ or $\frac{2}{5}$ or 0.4.

d) Eight cereals have no more than 3 gram of sugar, so the probability of selecting a cereal with no more than 3 g sugar is $\frac{8}{15}$ or 0.53.

The Law of Large Numbers

We have established that in theory, the probability of getting a head when you flip a coin is 50% or $\frac{1}{2}$. This is the theoretical probability. But how many heads will you get if you flip a coin 10 times? Will you definitely get 5 heads? Try it. The actual results for this experiment may not be that 50% of the outcomes will be heads. The actual results of this experiment is the empirical probability. How many heads do you think you will get if you continue to flip a coin 100 or even 1000 times?

The **Law of Large Numbers** states that as an experiment is repeated many times, the empirical probability gets closer to the theoretical probability. In other words, if you flip a coin 100 times, you are likely to get close to 50 heads, and if you flip a coin 1000 times it is very likely that you will have close to 500 heads.

Practice Exercise: Probability and the Law of Large Numbers

3a) What is the theoretical probability that the result of one coin toss is heads?

b) A computer program that does a coin flipping simulation gave the following results:

# of flips	# of heads	# of tails
10	7	3
100	56	44
1000	492	508

Use the empirical probabilities from the results above to explain how the results illustrate the Law of Large Numbers. Your explanation should include the theoretical probabilities as well.

3.7 SUMMARY

1. A twelve-sided die (with sides that are numbered from 1 to 12) is rolled once. Find the probability that it lands on a number that is:

a) greater than 6 ____________________________

b) a multiple of 5 ____________________________

c) an even number ____________________________

d) at most 7 ____________________________

2. A ___________________ can be used to list the sample space.

3. The _______________________principle can be used to calculate size of a large sample space without listing all possible outcomes. This principle states that if there are m ways to do one step of an experiment and n ways to do the second step, there are $m \cdot n$ ways to do both steps

4. State True/False for the following statements and **EXPLAIN** briefly why it is true or false.

a) Probability can be greater than 1.

b) As number of trials of an experiment increases, the empirical probability increases.

c) If there are 4 different shirts and 5 different pants to choose from to make an outfit, there are 20 different possible outfits consisting of a shirt and pants.

d) If there are 12 red balls and 16 green balls in an urn, the probability of picking a red ball is $\frac{12}{16}$.

3.8 Applications

In this section, we will solve applications by using some of the important concepts we have discussed in this chapter, including solving applications using systems of equations, as well as applications in statistics and probability.

Using a System of Equations to Solve Applications

Practice Exercise: Solving Applications by using Systems of Equations

For exercises 1 – 4, solve each by using a system of equations.

1) A snack of 16 peanuts and 5 cashews contains 135 calories. Another snack of 20 peanuts and 25 cashews contains 405 calories. How many calories are in each peanut and each cashew?

2) Two clubs are buying tickets for a movie. Twelve adult tickets and 10 student tickets cost $160. Eight adult tickets and fifteen student tickets cost $152.50. What is the price of an adult ticket and a student ticket?

3) Tuition at a community college is $450 plus $135 per credit. A local non-degree granting vocational school charges $300 for tuition, plus $150 per credit. Let x represent the number of credits that a student plans to take.

- **a)** Write a formula for the tuition at the community college.
- **b)** Write a formula for the tuition at the non-degree vocational school.
- **c)** At how many credits will the tuition be the same at both the community college and the vocational school?
- **d)** Your friend plans to take 12 credits this semester, and is only interested in saving money. Which school would you recommend to her: the community college or the vocational school? Explain your reasoning.

4) The cost of renting a car at Axis is $135 per week plus $0.35 per mile.
The cost of renting a car at Histz is $100 per week plus $0.55 per mile.

On graph paper:

- **a)** For each company, write an equation that represents the cost of renting a car.
- **b)** For each company, construct an input-output table with input, x, as the number of miles driven in one week. For the domain, use values from 0 to 500 in steps of 100.
- **c)** Use the ordered pairs from each table to graph each line on the same graph on graph paper. Label each line with its equation, and ***estimate*** the solution for this system.
- **d)** Find the number of miles that would result in the same rental cost at both companies.

Statistics Application

5) To be considered a rainforest, annual rainfall in an area must be 75 inches at a minimum, and most rainforests get over 100 inches of rain every year. The average amount of rainfall in the Amazon rainforest was recorded last week. The results are in the table below:

Days of the week	Amount of rain in inches
Sunday	0.5
Monday	2
Tuesday	3.5
Wednesday	1.5
Thursday	2
Friday	2
Saturday	0.5

a) Name the qualitative variable.____

b) Name the quantitative variable._______________________________________

c) Create a frequency distribution table and histogram for the amount of rain in inches.

d) Find the mean, median and mode for the amount of rain.

Probability Application

6) Raemin wants to know the outcomes if he tosses a die and then tosses a coin.

a) Draw a tree diagram to illustrate the sample space.

b) List the sample space.

c) How can you use the Fundamental Counting Principle to find the number of outcomes in the sample space?

7) Good passwords can consist of three letters followed by 3 digits. If repetition is allowed, how many different passwords can be created?

Answers

Section 1.1: Practice Exercises

1) \$2.45
2) 2.45 + 0.19 (4 – 3) = \$2.64
3) 2.45 + 0.19 (10 – 3) = \$3.78

Section 1.1: Summary

1. Variable
2. Understand the problem
3. Numerical
4. Symbolic

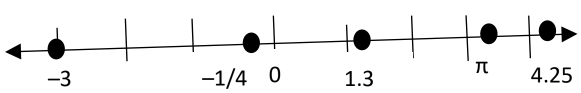
5.

x	$y = x - 3$
–2	–5
–1	– 4
0	–3
1	–2
2	–1

6. **a)** \$35
b) \$35
c) 0.02 (2000) = \$40

Section 1.2: Practice Exercises

1) True, all the natural numbers are in whole numbers
2) True, any integer can be written in the form $\frac{a}{b}$ as a rational number
3) False, π cannot be written as the quotient of two integers
4) False, natural numbers do not have negative integers
5) False, natural numbers do not include 0
6)



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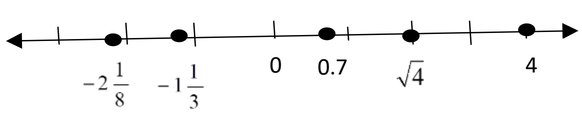

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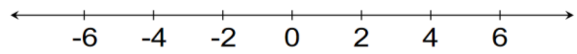


9) numerical coefficient: –4, no constant
10) numerical coefficients: 5, –1, constant: –3
11) numerical coefficient: 2, constant: π
12) numerical coefficient: 4, constant: $-\sqrt{5}$
13) $n - 8$
14) $q - 8$
15) $\frac{1}{5}n$
16) $x - 2$
17) $\frac{x}{6}$
18) $1 + 2x$
19) $5(2x)$
20) $y = \frac{1}{2}x$
21) $y = 2x - 5$
22) $\frac{x}{3} - 3 = y$
23) $y = 550x + 300$
24) $y = 25 - 2.50x$
25) $y = 0.04\,x$
26) $y = 250 + 0.95x$
27) **a)** $y = 5 - 0.10x$
b)

Number of copies made, x	**Balance remaining on the copy card, y**
0	5
10	4
20	3
30	2
40	1
50	0

c) \$4

d) 50 copies

28) **a)** $y = 0.01x$
b) $y = 0.015x$
c) \$10
d) \$45

Section 1.2: Summary

1. The set of **Real numbers** includes __rational__ and __irrational__ numbers.

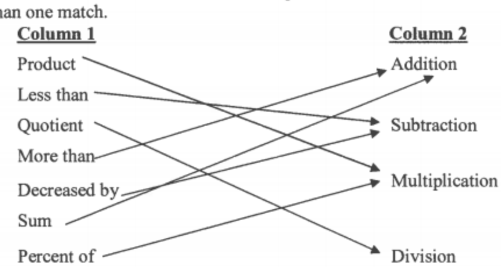
- __Rational__ numbers can be written as a quotient of two integers.
  - ✓ __Natural__ numbers → {1, 2, 3, 4,…}
  - ✓ __Whole__ numbers → {0, 1, 2, 3, 4,…}
  - ✓ __Integers__ → {…-3, -2, -1, 0, 1, 2, 3,…}
- __Irrational__ numbers cannot be written as a quotient of two integers

2. In the expression $2x-5$:

- __x__ is the variable
- __2__ is the numerical coefficient of _x_
- __-5__ is the constant.

3.

Match the word in Column 1 with the operation in Column 2. Some operations have more than one match.



4. **a)** $y = 4x + 1$ **b)**

x	y
1	5
2	9
3	13
4	17

5. **a)** $4 - x$
b) $3x$
c) $\frac{x}{8}$ *or* $x \div 8$
d) $2x–5$

Section 1.3: Practice Exercises

A) (2, 10) **B)** (2, 9) **C)** (2, 8) **D)** (2, 5) **E)** (2, 2) **F)** (0, 6) **G)** (–6, 7) **H)** (–10, 0)

I) (–4, –3) **J)** (0, 0) **K)** $\left(\frac{1}{2}, -2\frac{1}{2}\right)$ **L)** (0, –8) **M)** (2, –9) **N)** (6, –2) **P)** (7, 0)

1) After 3 months, the total amount in the savings account is $100
2) At start, the total amount in the saving account is $10
3) After a year, the total amount in the savings account is $370

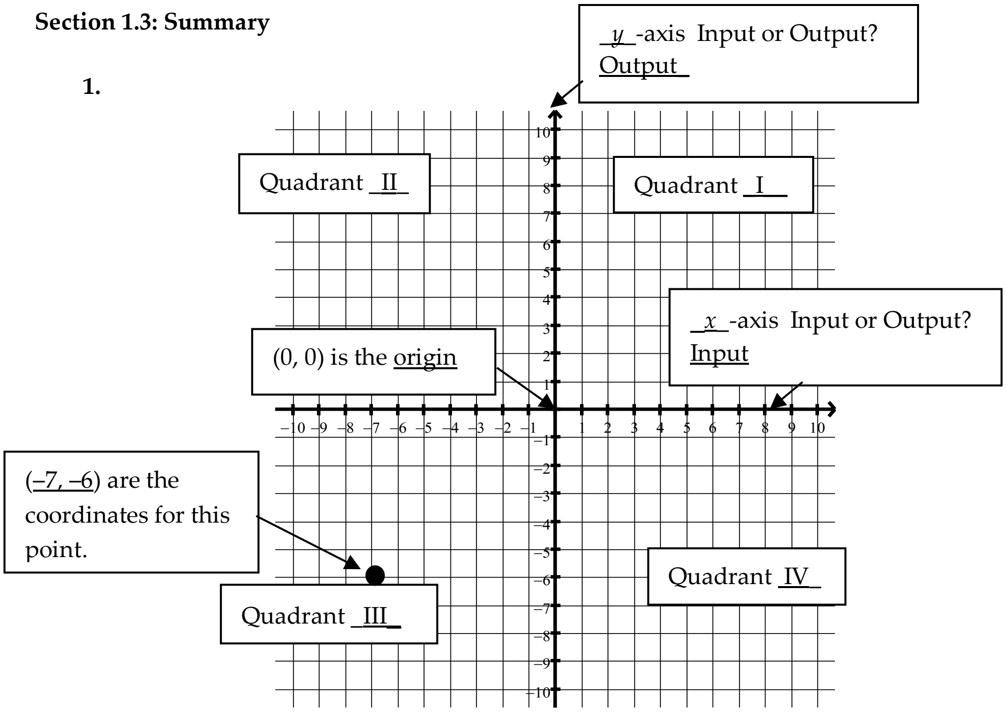
Example 3: Numerically

# of Hours	Contractor's cost	Ordered Pairs (x, y)
0	450	(0, 450)
1	488	(1, 488)
2	526	(2, 526)
3	564	(3, 564)
4	602	(4, 602)
5	640	(5, 640)

4) After 2 hours of labor, the total cost of the contractor's job is $526
5) After 5 hours of labor, the total cost of the contractor's job is $640

Section 1.3: Summary

1.



2. An ordered pair (x, y) contains two real numbers that describe a point on a coordinate plane.

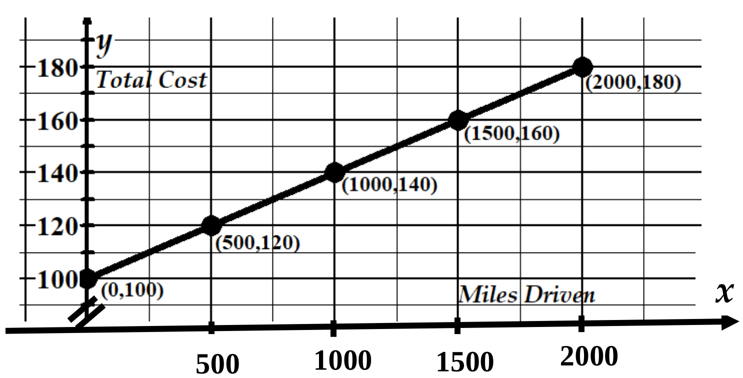
- The first number, x, describe the horizontal distance from the origin, along the x-axis.
- The second number, y, describes the vertical distance from the origin, along the y-axis.

3. The scale on the axis is the amount the spaces represent between tick marks on the axis. For example, the scale of the horizontal axis below is ten.

4. The input-output table below shows the cost for driving a van x-miles. A reasonable horizontal (x-) axis scale for the graph is 500. If you use a break, a reasonable vertical (y-) axis scale is 20.

***x* - (input)** **Miles Driven**	***y* - (output)** **Total Cost ($)**	**Ordered Pairs** **(*x*, *y*)**
0	100	(0, 100)
500	120	(500, 120)
1000	140	(1000, 140)
1500	160	(1500, 160)
2000	180	(2000, 180)

5.



6. (500, 120) means that the cost of driving 500 miles, is $120

7. At the beginning without driving any miles (0 miles), the cost is $100.

Section 1.4: Practice Exercises

1) a) $8y^3$ **b)** $9y^2$

2) a) $2y^3$ **b)** 48

3) a) x^6 **b)** -24

4) a) x^6 **b)** $75x^2$

5) a) $64x^9y^3$ **b)** a^3b^5

6) a) $4a^6b^2$ **b)** $-6x^3y^7$

7) a) $9x^8y^6$ **b)** $75x^8$

8) y^4

9) $\frac{3}{8}$

10) $\frac{3a}{c}$

11) $-\frac{6}{5y}$

12) $54x^4$

13) $9x^2$

14) $-\frac{9}{x^2}$

15) $\frac{x^2}{y^4}$

16) 5

17) -5

18) 5

19) 25

20) 0.6

21) $\frac{2}{3}$

22) 6

23) 6

Section 1.4: Summary

1.

Fill in the blanks

m is called the exponent

a^m

a is called the base

2. **a)** $\frac{3x}{y^2}$

b) $-10a^6b^3$

c) a^6

3.

$(ab)^3 = a^3b^3$ and $\left(\frac{a}{b}\right)^4 = \frac{a^4}{b^4}$

$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$ and $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

4.

I. $(-6xy)^2 =$

a. $36xy$ b. $-36x^2y^2$ c. $36x^2y^2$ d. $6x^2y^2$

II. $\frac{\sqrt{48}}{\sqrt{3}} =$

a. 16 b. 4 c. 6 d. none of these

Section 1.5: Practice Exercises

1) -25
2) 8
3) 12
4) 0
5) 10
6) 7
7) 4
8) 44
9) -5
10) -54
11) $(7 + 2)$, Associative property of addition
12) (4), Commutative property of multiplication
13) $(2 \cdot 3)$, Associative property of multiplication
14) (x), Commutative property of addition
15) $(x - 5)$, Commutative property of multiplication
16) $2x + 6y - 8z$
17) $-2x + 19$
18) 11
19) $4x^2,\ -2xy,\ -y,\ 1$
20) $4, x, (x - 3)$
21) $20x$
22) $9a - 6b$
23) $-5x^3 + 8x^2 + 10x$
24) $-2x^2y$
25) $13x - 27$
26) $4x$
27) $2x - 9$
28) $-8x - 9$
29) $7x^2 + 7x + 18$
30) $y^2 - 3y + 7$
31) $-3y$
32) $-3x + 2$
33) $\frac{5x}{3} + 5$
34) $5x$
35) $2x + \frac{16}{3}$

Section 1.5: Summary

1. P - parentheses or other grouping symbols, i.e. brackets, absolute value, square roots

E - exponents

$\left.\begin{matrix}\text{M}\\ \text{D}\end{matrix}\right\}$ multiplication or division from left to right

$\left.\begin{matrix}\text{A}\\ \text{S}\end{matrix}\right\}$ addition or subtraction from left to right

2. **a)** 44
b) 8

3.

Factors vs. Terms The difference between factors and terms is that factors are multiplied (added or multiplied) and terms are added (added or multiplied).

Example:
The expression $2x - 3xy$ has two terms. The first term is $2x$.
The factors of the first term are 2 and x. The second term is $-3xy$.
The factors of the second term are: -3, x and y.

4.

To **combine like terms**, add (add, multiply, keep) the coefficients and keep (add, multiply, keep) the variable.

Example: a) $-5x + 10y + 2x + 3y = -3x + 13y$ **b)** $x^2 + 2x + 7x^2 - 5x = 8x^2 - 3x$

5. **Properties of Real Numbers**

Associative Property (Regroup): $a+(b+c)=(a+\underline{b})+\underline{c}$ **OR** $a(b\cdot c)=(\underline{a}\cdot b)\cdot \underline{c}$
Commutative Property (Reorder): $a+b=\underline{b+a}$ **and** $ab=\underline{ba}$
6. **Distributive Property (Multiplication Across Addition):** $a(b+c)=\underline{ab}+\underline{ac}$

- Fraction with a **single term** in numerator and denominator – cancel common factors.
  $$\frac{-18ab}{12bc}=\frac{-3a}{2c}$$
- Fractions with **more than one term in the numerator and one term in the denominator** – use the distributive property to divide each term by the denominator; then cancel common factors in each term. $\frac{16x-10}{4}=\frac{16x}{4}-\frac{10}{4}=4x-\frac{5}{2}$

Section 1.6: Practice Exercises

1)

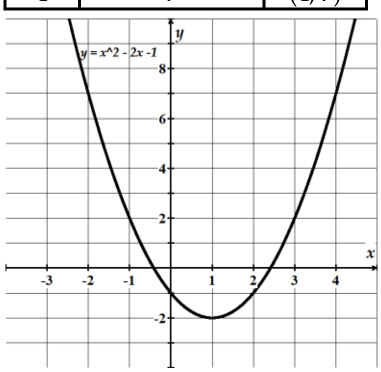
Polynomial	Name	Leading coefficient	Constant term	Degree
$2x^5-x^2+3$	trinomial	2	3	5
$5-x^2$	binomial	–1	5	2
$2x-1$	binomial	2	–1	1
$-8x^3$	monomial	–8	0 (or none)	3

2) $7x^2-4x-1$
3) $4x+24$
4) $-x^2+4x$
5) $-4x^3+x^2-2$
6) $2x^3-6x^2-2x$
7) $2x^4-6x^2-8x$
8) $-6x^5$
9) $-x+1$
10) x^2+1
11) $x^2+8x+15$
12) $x^2-8x+15$
13) $x^2+2x-15$
14) $x^2-2x-15$
15) x^2-1
16) x^2+6x+9
17) $2x^2-11x+12$
18) $6x^2+5x-4$
19) $4x^2-12x+9$
20) $4x^2-9$
21) 14
22) 18
23) 6
24) –25
25) 36
26) –31

27) ***a)***

x	$y=x^2-2x-1$	(x, y)
–2	7	(–2, 7)
–1	2	(–1, 2)
0	–1	(0, –1)
1	–2	(1, –2)
2	–1	(2, –1)
3	2	(3, 2)
4	7	(4, 7)

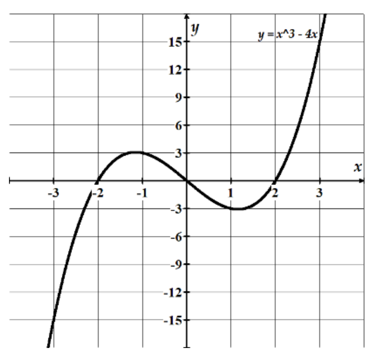
b)



28) ***a)***

x	$y=x^3-4x$	(x, y)
–3	–15	(–3, –15)
–2	0	(–2, 0)
–1	3	(–1, 3)
0	0	(0, 0)
1	–3	(1, –3)
2	0	(2, 0)
3	15	(3, 15)

b)



Section 1.6: Summary

1. A **polynomial** is an expression containing one or more terms being ___added__ or subtracted. The exponents on the variables in each term must be ___positive____ integers.

- **Monomial** - a polynomial expression containing __one___ term.
- **Binomial** - a polynomial expression containing __two___ terms.
- **Trinomial** - a polynomial expression containing __three___ terms.

Example: The polynomial $-5x^3 + x - 1$ has _three__ terms. The leading term is $\underline{-5x^3}$ and the coefficient of the leading term is $\underline{-5}$. The constant term is $\underline{-1}$.

2. **To add or subtract polynomials**, combine __like___ terms.

Example:
$$(4x^2 - 3x + 2) - (x^2 - 2x + 4)$$
$$= 4x^2 - 3x + 2 - x^2 \underline{+} 2x \underline{-} 4 = \underline{3x^2 - x - 2}$$

3. **To multiply a monomial by a polynomial** use the __distributive___ property.

Example: $-3x^4(x^3 - 20x + 10) = \underline{-3x^7 + 60x^5 - 30x^4}$

4. **To multiply binomials** use distribution twice or FOIL

Example:
$$(x+4)(x-5) = \underset{F}{x^2} \underset{O}{- 5x} \underset{I}{+ 4x} \underset{L}{- 20} = \underline{x^2 - x - 20}$$

5. To **evaluate an algebraic expression** for a given value means to __substitute___ the value of the variable each time the variable occurs.

Example: Evaluate $-x^2 + 3x - 2$, for $x = -2$

Solution: $-(\underline{-2})^2 + 3(\underline{-2}) - 2 = -\underline{4} + \underline{(-6)} - 2 = \underline{-10 - 2 = -12}$

Replace each x with -2
!!Remember to put the (–2) in parentheses!!

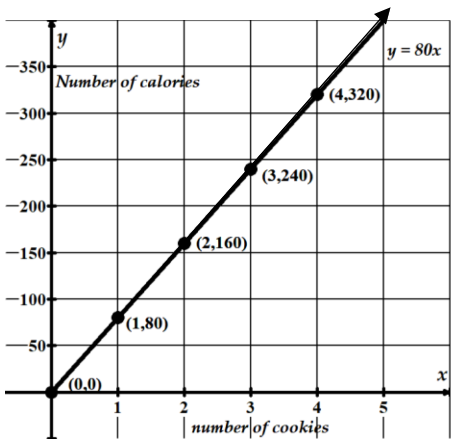
Follow order of operations to simplify.

Section 1.7: Practice Exercises

1) Function. Each input is associated with a single output.
2) Function, each student has his/her own birthday.
3) Not a function. Input (Sept 1) can be more than one student's birthday.

4) $y = 80x$. Yes, it is a function

Number of cookies, x	Number of calories, y	(x, y)
1	80	(1, 80)
2	160	(2, 160)
3	240	(3, 240)
4	320	(4, 320)



5) **i)** w = input
C = output
ii) w = the number of ounces a package weights
C = cost to mail a package
iii) is a function. $C(w) = 0.2w$

6) **i)** w = input
C = output
ii) w = the number of lbs for ham
C = cost of buying
iii) is a function. $C(w) = 5.99w$

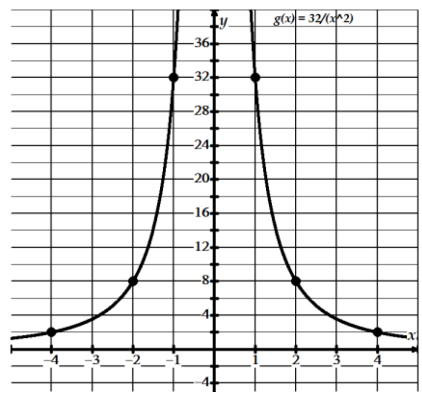
7) **i)** n = input
C = output
ii) n = the number of copies
C = cost of making copies
iii) is a function. $C(n) = 0.1n$

8) 0
9) 1
10) – 27
11) 8
12) 6
13) When the student takes 12 credits, the tuition is \$5400
14) When the student takes 15 credits, the tuition is \$6750

15)a)

x	$g(x) = \frac{32}{x^2}$	$(x, g(x))$
– 4	2	(– 4, 2)
– 2	8	(– 2, 8)
–1	32	(– 1, 32)
0	Not defined (error)	undefined
1	32	(1, 32)
2	8	(2, 8)
4	2	(4, 2)

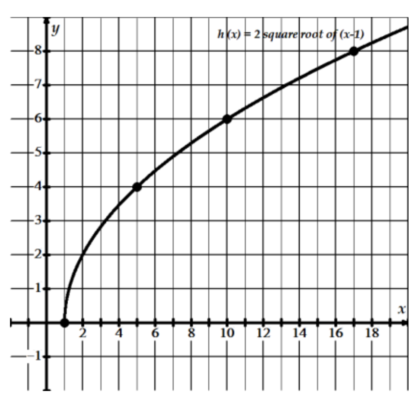
15) a)



15 b)

x	$h(x) = 2\sqrt{x-1}$	$(x, h(x))$
1	0	(1, 0)
2	2	(2, 2)
5	4	(5, 4)
10	6	(10, 6)
17	8	(17, 8)

15 b)



16) $g(-4)=2$

17) $g(4)=2$

18) $h(17)=8$

19) $h(82)=18$

20) $g(-1)=32,\ g(1)=32$,So $x=-1$ and $x=1$

21) $g(-4)=2,\ g(4)=2$,So $x=-4$ and $x=4$

22) $h(2)=2$, so $x=2$

23) $g(1)=32$

24) $h(1)=0$

25) **a)** When 0 rides are taken, the amount remaining on the Metro Pass is $25.
b) When 10 rides are taken, the amount remaining on the Metro Pass is $0.

Section 1.7: Summary

1. y is a $\boxed{\textbf{function}}$ of x, if for each _input_ (*input* or *output*), there is only one _output (*input* or *output*)
2. When verifying if *y is a function of x,* you must either write:

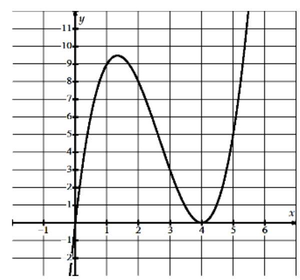
Yes, y is a function of x because for each _x_ (x or y), there is only one _y_ (x or y) OR No, y is not a function of x because for some _x_(x or y), there is more than one _y_(x or y).

In function notation, *y is a function of x* is written $y=f(x)$.

3. The __domain____ of a function is the set of all allowable inputs.
The ____range___ of a function is the set of all outputs
4. **a)** The table below does not represent a function because the input of $x = 15$ has two outputs.

x	y	(x, y)
10	4	(10, 4)
15	8	(15, 8)
0	−7	(0, –7)
15	−12	(15, –12)
22	20	(22, 20)
35	8	(35, 8)

b) The graph below represents a function because any vertical line crosses the graph in exactly one point



5. We use the _vertical line___ test to decide if a graph is a function.
6. $f(2) = 125$. The cost of two admission tickets to an amusement park is $125.
7. If $f(x)=x+1$,

a) Evaluate $f(3)$.

$f(3) = 3+1$
$f(3) = 4$

b) Evaluate $f(-1)$.

$f(-1) = -1+1$
$f(-1) = 0$

Section 2.1: Practice Exercises

1) Expression
2) Equation
3) Equation
4) Expression
5) Equation
6) When $x=-4$, we have $3(-4-1)=-15$, , since $-15=-15$, Yes, $x=-4$ is a solution
7) When $x=0$, we have $3(0-1)=-3$, , since $-3=-3$, Yes, $x=0$ is a solution
8) When $x=-4$, we have $2(-4-1)=-10$, , since $-10\neq-15$, No, $x=-4$ is not a solution
9) When $x=-4$, we have $3(-4-1)=-15$, since $-15=-15$, Yes, $x=-4$ is a solution

10) When $x = -3$, we have $3(-3-1) = -12$, also $5(-3+1) = -10$ Since $-12 \neq -10$, No, $x = -3$ is not a solution

11) $x = -5 + 1 = -4$. Check: $(-4) - 1 = -5$

12) $x = -15 - 7 = -22$. Check: $7 + (-22) = -15$

13) $x = -3$. Check: $2(-3) - 9 = -15$

14) $x = 8$. Check: $7 = 15 - 8$

15) $x = -\dfrac{11}{5}$. Check: $5\left(-\dfrac{11}{5}\right) - 1 = -12$

16) $x = -6$. Check: $-4(-6) = 24$

17) $x = \dfrac{10}{3}$. Check: $-3\left(\dfrac{10}{3}\right) = -10$

18) $x = -2$. Check: $8 - 2(-2) = 12$

19) $x = 22$. Check: $7 - (22) = -15$

20) $x = -\dfrac{25}{3}$. Check: $10 + 3\left(-\dfrac{25}{3}\right) = -15$

21) $f(x) = 3x - 4$

22) $f(x) = 2x$

23) $f(x) = \dfrac{x}{6}$

24) **i)** c: input, cost of the item
t: output, sales tax
ii) $t(c) = 0.07c$

25) **i)** n: input number of rides,
R: output, remaining value
ii) $R(n) = 25 - 3.50\,n$

26) **a)** $x = 0$
b) $x = 4$
c) $x = 5$
d) x is between 1 and 2

27) **a)** $x = 1$
b) $x = 2$
c) x is between –1 and 0

28) **a)** $f(2) = 2(2) - 4 = 0$, check the answer
b) $f(0) = 2(0) - 4 = -4$, check the answer
c) $2x - 4 = -4$, $x = 0$, check the answer
d) $2x - 4 = -5$, $2x = -1$, $x = -\dfrac{1}{2}$, check the answer

29) **a)** $x = -1$
b) $x \approx 0.3$
c) $x = 1$
d) $x = 0$

30) **a)** $x = -2$
b) $x = 2$
c) $x = 0$
d) $x = 4$

Section 2.1: Summary

1) **a)** equations have equals signs.
b) expressions do not have equals signs.
c) equations are solved for the value(s) of the variable.

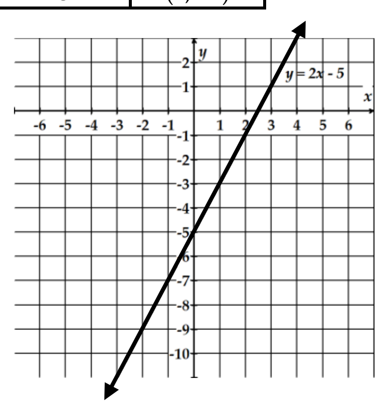
2) To solve $2x - 5 = -3$ $\boxed{\textbf{numerically}}$:

Step 1: Create an input-output table for the rule y = 2x–5 (or examine a given table).
Step 2: Look in the output (input or output) column for –3.
Step 3: The solution is the corresponding input (input or output) value.
Step 4: Write solutions in the form x = value. For this example the solution is $x = 1$.

x	$y = 2x - 5$	(x, y)
–3	–11	(–3, –11)
–2	–9	(–2, –9)
–1	–7	(–1, –7)
0	–5	(0, –5)
1	–3	(1, –3)

3) To solve $2x - 5 = -9$ $\boxed{\textbf{graphically}}$:

Step 1: Sketch a graph for the rule y = 2x–5
Step 2: Look on the y-axis for – 9.
Step 3: The solution is the corresponding input (input or output) value.
Step 4: Write solutions in the form x = value. For this example the solution is $x = -2$.



4) To solve an equation **symbolically/algebraically**, use the addition and multiplication properties of equality, which state you can __add__ or __multiply__ the same number to both sides of an equation to produce an equivalent equation.

Example: Solve $2x-5=-3$ symbolically.

$$2x-5\ +\ 5\ =-3+5$$
$$\frac{2x}{2}=\frac{2}{2}$$
$$x=1$$

Section 2.2: Practice Exercises

1) Rational equation
2) Radical equation
3) Cubic equation
4) Linear equation
5) Quadratic equation
6) Linear equation
7) **a)** $x=0$
b) $x=-2$
c) $x=-1, x=1, x=2$
d) $x=3$
8) **a)** $x=0$
b) $x\approx-1$
c) no solutions
d) $x=8$
9) **a)** $x=-2$
b) $x=-1, x=1, x=2$
c) $x\approx 2.6$
d) $x=3$
10) $y=-2$ (check the answer)
11) $y=-4$ (check the answer)
12) $n=-9$ (check the answer)
13) $y=3$ (check the answer)
14) $x=3$ (check the answer)
15) $y=-5$ (check the answer)
16) $y=-\frac{5}{3}$ (check the answer)
17) $y=\frac{11}{2}$ (check the answer)
18) $y=21$ (check the answer)
19) $x=-3$ (check the answer)
20) $4(x-1)=12,\quad x=4$ (check the answer)
21) $5+3x=31,\quad x=\frac{26}{3}$ (check the answer)

Section 2.2: Summary

1. $x^2-4x-5=-5$ has __2__ solution(s). List the solution(s): __$x=0, x=4$__
$x^2-4x-5=-9$ has __1__ solution(s). List the solution(s): __$x=2$__
$x^2-4x-5=-10$ has __no__ solutions.

Example:

x	$y=x^3-4x+1$
−3	−14
−2	1
−1	4
0	1
1	−2
2	1
3	16

$x^3-4x+1=16$ has __1__ solution(s). List the solution(s).
__$x=3$__

$x^3-4x+1=1$ has __3__ solution(s). List the solution(s).

__$x=-2,\ \ x=0,\ \ x=2$__

The solution to the equation $x^3-4x+1=-10$ would fall between the numbers __−3__ and __−2__.

2. To solve an equation containing parentheses, the __distributive__ property must be applied to remove them.

Example: Circle the correct solution to the equation $8-3(x+1)=15$. Show the check.

Solution 1:

$8-3(x+1)=15$
$5(x+1)=15$
$5x+5=15$
$5x=10$
$x=2$

Solution 2:

$8-3(x+1)=15$
$8-3x-3=15$
$-3x+5=15$
$-3x=10$
$x=-\frac{10}{3}$

Check:

$8-3(-\frac{10}{3}+1)\overset{?}{=}15$
$8-3\left(-\frac{10}{3}+\frac{3}{3}\right)\overset{?}{=}15$
$8-3\left(-\frac{7}{3}\right)\overset{?}{=}15$
$8+7\overset{?}{=}15$
$15 = 15$ √True

Section 2.3: Practice Exercises

1) $x = 3$ (check the answer)
2) $a = 0$ (check the answer)
3) $x = 26$ (check the answer)
4) $x = -27$ (check the answer)
5) $x = 33$ (check the answer)
6) $x = 21$ (check the answer)
7) $x = 5$ (check the answer)
8) $x = 0$ (check the answer)
9) $x=\frac{9}{5}$ (check the answer)
10) $x = -2$ (check the answer)
11) $x = 11$ (check the answer)
12) $y = -9$ (check the answer)
13) $x = 5$ (check the answer)
14) $y=\frac{15}{2}$ (check the answer)
15) $x = -3$ (check the answer)
16) $x = 3$ (check the answer)
17) $y = 2$ (check the answer)
18) $x = -7$ (check the answer)
19) $x = 7$ (check the answer)

Section 2.3: Summary

1.

- Apply the __distributive__ property to expressions with parentheses
- __Combine__ like terms.
- Get all the terms with the variable to be solved for onto one side of the equation and constant terms to the other side of the equation.
- Add like terms yet again.
- Solve for the variable using the multiplication property of equations.

2. When solving an equation with variables on both sides, __simplify__ the expressions on each side before moving variables to one side and constants to other.

Example: Describe the steps to solve the equation, $85 + 5(x - 11) = 100 - 2x$.

$85+5(x-11)=100-2x$ -- Given problem.
$85+5x-55=100-2x$ --__Distributed the 5__
$5x+30=100-2x$ --__Combined like terms__
$7x+30=100$ --__Moved variable terms to one side__
$7x=70$ --__Moved constant terms to opposite side__
$x=10$ --__Divided by 7 to isolate x__

Check this solution

$85+5(x-11)=100-2x$

$85+5(\underline{10}-11)\stackrel{?}{=}100-2(\underline{10})$

$\underline{85+5(-1)}\stackrel{?}{=}\underline{100-20}$

$80=80\ \surd$

Section 2.4: Practice Exercises

1) $x = -28$ (check the answer)
2) $x = 12$ (check the answer)
3) $x = 6$ (check the answer)
4) $x = 21$ (check the answer)
5) $x = -135$ (check the answer)
6) $x = 24$ (check the answer)
7) $x = 20$ (check the answer)
8) $x = \frac{21}{5}$ (check the answer)
9) $n = \frac{25}{4} = 6.25$ (check the answer)
10) $x = 36$ (check the answer)
11) $x = 5$ (check the answer)
12) $x = 9$ (check the answer)

Section 2.4: Summary

1. <u>STEPS for Solving Linear Equations Symbolically</u>

- Clear fractions, if applicable.
- Apply the ______distributive________ property to expressions with parentheses.
- Combine__ like terms.
- Get all the terms with the variable to be solved for onto one side of the equation and constant terms to the other side of the equation.
- Add like terms yet again.
- Solve for the variable using the multiplication property of equations.

<u>Example</u>:

To clear denominators, the equation should be multiplied by the __least common denominator___.

a) For $\frac{1}{3}x+4=7$ multiply each term on both sides by __3__.

b) For $5-\frac{2}{5}x=\frac{1}{3}$ multiply each term on both sides by __15_ .

c) For $\frac{7}{2}=\frac{1}{2}x-\frac{1}{6}$ multiply each term on both sides by __6__ .

2. Cross-multiplication can be used to solve a $\boxed{\text{proportion}}$

$$\frac{a}{b}=\frac{c}{d} \rightarrow \underline{ad} = \underline{bc} \text{ where } b \neq \underline{0} \text{ and } d \neq \underline{0}$$

This equation is a proportion.

$$\frac{5x-1}{9}=\frac{x+7}{3}$$

Denominators are cleared here by cross-multiplication
*Note the need for parentheses.

$$\begin{aligned} 3(5x-1)&=9(x+7)\\ 15x-3&=9x+63\\ 6x-3&=63\\ 6x&=66\\ x&=11 \end{aligned}$$

vs.

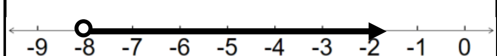
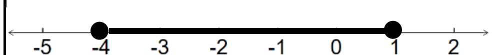
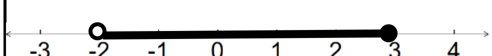
This equation is not a proportion.

$$\frac{5}{9}x-1=\frac{x}{3}+7$$

Cross-multiplication cannot be applied.
To clear denominators, multiply by the LCD, 9.

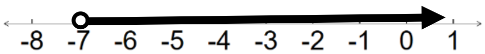
$$\begin{aligned} 9\left(\frac{5}{9}x-1\right)&=9\left(\frac{x}{3}+7\right)\\ 5x-9&=3x+63\\ 2x-9&=63\\ 2x&=72\\ x&=36 \end{aligned}$$

Section 2.5: Practice Exercises

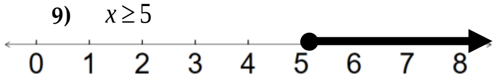
	NUMBER LINE	SYMBOLS	WORDS
1)		$x \le -8$	x is less than or equal to – 8
2)		$x > -8$	x is greater than – 8
3)		$-4 \le x \le 1$	x is between – 4 and 1, inclusive
4)		$x > 12$	x is greater than 12
5)		$-1 \le x < 0$	x is between –1 and 0, including –1
6)		$-2 < x \le 3$	x is between –2 and 3, including 3.
7)		$x < 0$	x is less than 0

(Be sure also to check your answers for 8-15)

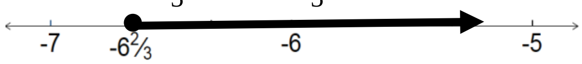
8) $x > -7$



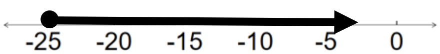
9) $x \geq 5$



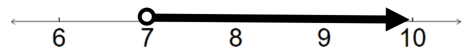
10) $x \geq -\dfrac{20}{3}$ or $x \geq -6\dfrac{2}{3}$



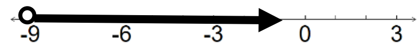
11) $x \geq -25$



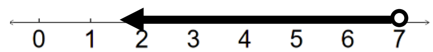
12) $7 < x$ or $x > 7$



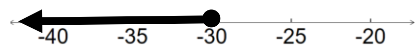
13) $x > -9$



14) $7 > x$ or $x < 7$



15) $x \leq -30$



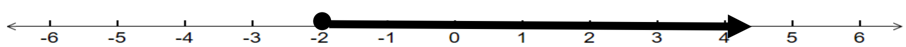
Section 2.5: Summary

1. An __inequality__ is a statement that one quantity is greater than another. The four inequality symbols are: $<, >, \leq, \geq$.

<u>**Examples**</u>**:**

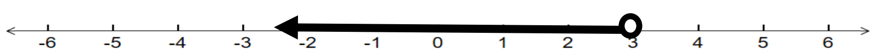
- In symbols, the set of numbers greater than or equal to -2 is written: $\underline{x \geq -2}$

On a number line:



- In symbols, the set of numbers less than 3 is written: $\underline{x < 3}$

On a number line:

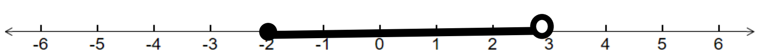


2. A __compound__ inequality consists of two inequalities combined.

<u>**Example**</u>**:**

- In symbols, the set of numbers between -2 and 3, including -2, is written: $\underline{-2 \leq x < 3}$

On a number line:



3. To solve a linear inequality __reverse__ the inequality sign when dividing or multiplying by a negative number.

<u>**Examples**</u>**:** Circle the correct solution:

a) $-2x > 10$

i) $x > -5$ **ii)** $x < 5$ **iii)** $x < -5$ **iv)** $x \leq -5$

b) $3 - 3x \leq -12$

i) $x \geq -5$ **ii)** $x \leq 5$ **iii)** $x \geq 5$ **iv)** $x \leq -5$

Section 2.6: Practice Exercises

1) $b = \dfrac{2A}{h}$

2) $l = \dfrac{P-2w}{2} = \dfrac{P}{2} - w$

3) $y = \dfrac{2x-6}{3} = \dfrac{2}{3}x - 2$

4) $y = \dfrac{3x-6}{2} = \dfrac{3}{2}x - 3$

5) $r = \dfrac{C}{2\pi}$

6) $C_1 = \dfrac{C_2 V_2}{V_1}$

7) $r = \dfrac{A-P}{Pt} = \dfrac{A}{Pt} - \dfrac{1}{t}$

8) $y = \dfrac{-4x-9}{3} = -\dfrac{4}{3}x - 3$

Section 2.6: Summary

1. Use the word bank to fill in the blanks below

- ___Formulas___ are used frequently in everyday life, including geometry calculations, sports, and interest.
- A formula is an ___equation___.
- Solving for a variable in an equation means it is isolated and written ___in terms of___ the other variables.
- The ___distributive___ property can be applied to rewrite fractions with more than one term in the numerator.
- A subscripted variable represents a ___constant___ value, often an initial value of the variable.

Word Bank
distributive
equation
formulas
constant
in terms of

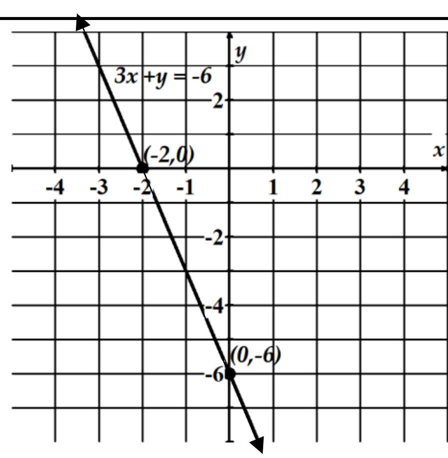
2. State what operation (add, subtract, multiply or divide) is needed to solve for y in each formula.

a) To solve $V = xyz$ for y, ___divide___ both sides by ___x___ and ___z___.

b) To solve $2x - 3y = 6$ for y, ___subtract___ $2x$ on both sides and ___divide___ by -3.

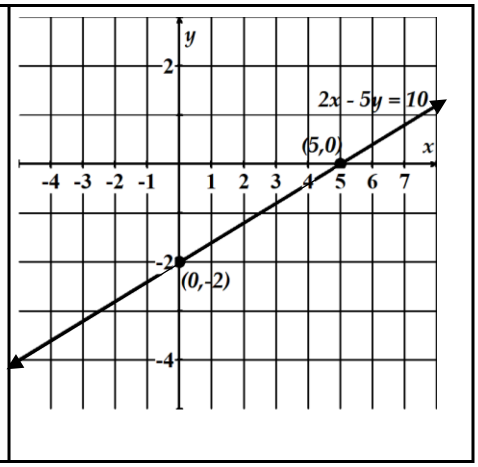
Section 2.7: Practice Exercises

1)

$3x + y = -6$ The x-intercept is $(-2, 0)$ The y-intercept is $(0, -6)$	

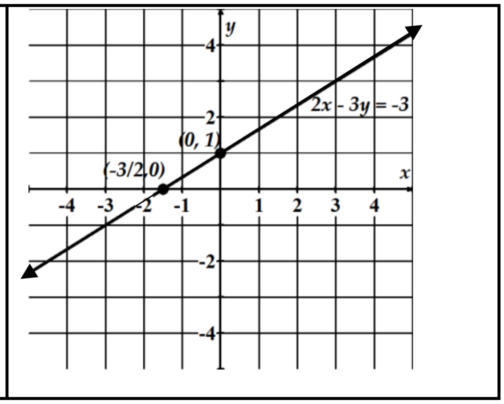
2)

$2x - 5y = 10$
The x-intercept is $(5, 0)$
The y-intercept is $(0, -2)$



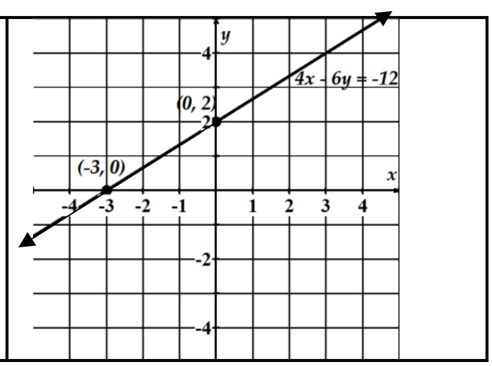
3)

$2x - 3y = -3$
The x-intercept is
$\left(-\frac{3}{2}, 0\right)$
The y-intercept is $(0, 1)$



4)

$4x - 6y = -12$
The x-intercept is $(-3, 0)$
The y-intercept is $(0, 2)$



5) **a)** $y = 40 - 5x$

b) To find the horizontal axis intercept, let $y = 0$:

$y = 40 - 5x$
$0 = 40 - 5x$ add $5x$ to both sides
$5x = 40$
$x = 8$

The horizontal axis intercept is $(8, 0)$. Since the x values are the number of movies he has seen, the meaning of this intercept is that when 8 movies are seen, the value of the card is \$0, so Julio can see 8 movies with the \$40 gift card.

c) To find the vertical axis intercept, let $x = 0$:

$y = 40 - 5(0)$
$y = 40$

The vertical axis intercept is $(0, 40)$. Since y represents the remaining value of the card, the meaning of this intercept is that when the card is worth \$40, Julio hasn't seen any movies yet.

6)

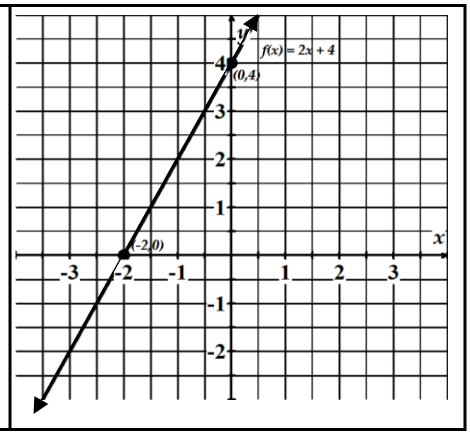
$f(x) = 2x + 4$

To find the x-intercept, let $f(x) = 0$

The x-intercept is (–2, 0)

To find the y-intercept, let $x = 0$

The y-intercept is (0, 4)



7)

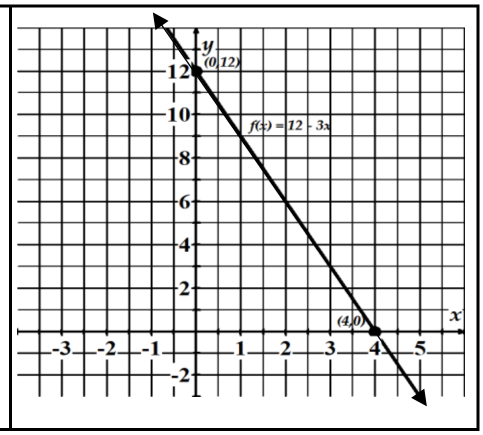
$f(x) = 12 - 3x$

To find the x-intercept, let $f(x) = 0$

The x-intercept is (4, 0)

To find the y-intercept, let $x = 0$

The y-intercept is (0, 12)



8)

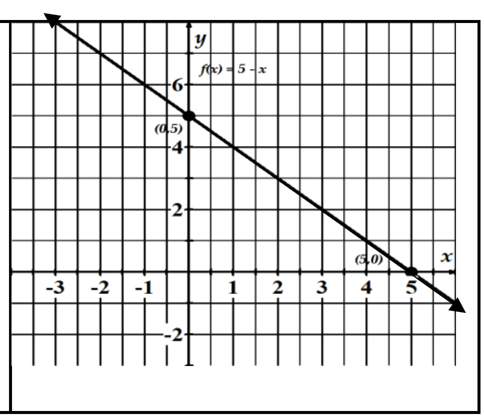
$f(x) = 5 - x$

To find the x-intercept, let $f(x) = 0$

The x-intercept is (5, 0)

To find the y-intercept, let $x = 0$

The y-intercept is (0, 5)



9)

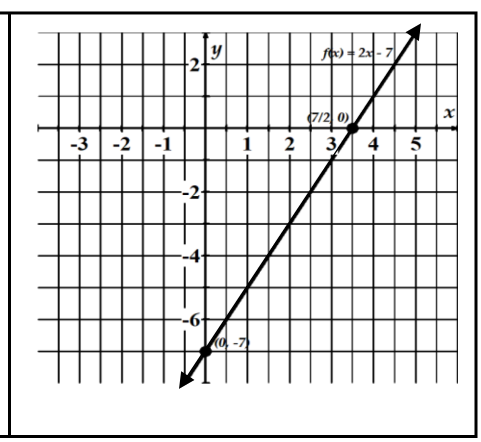
$f(x) = 2x - 7$

To find the x-intercept, let $f(x) = 0$

The x-intercept is $\left(\frac{7}{2}, 0\right)$

To find the y-intercept, let $x = 0$

The y-intercept is (0, – 7)



10)

$f(x) = 5x$

To find the x-intercept, let $f(x) = 0$

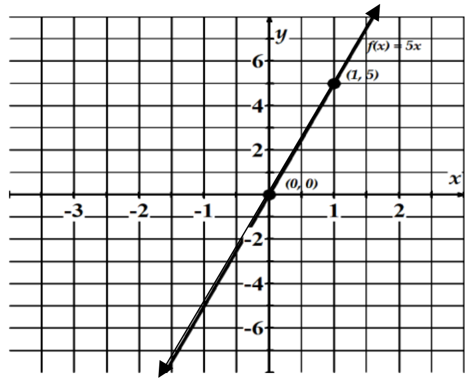
The x-intercept is (0, 0)

To find the y-intercept, let $x = 0$

The y-intercept is (0, 0).

We need another point to graph this line. If we let $x = 1$, then $f(1) = 5(1) = 5$

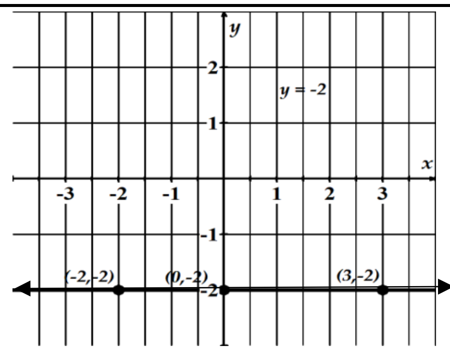
So we plot the point (1, 5).



11)

$y = -2$

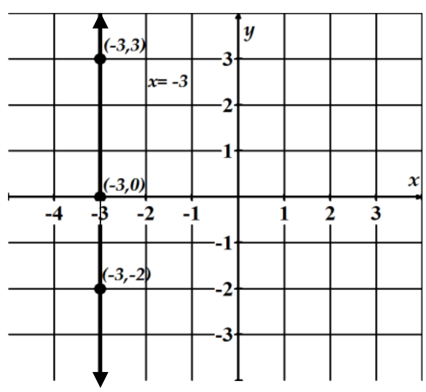
The horizontal line crosses the y-axis at $y = -2$, which is the equation of this line passing through (–2, –2).



12)

$x = -3$

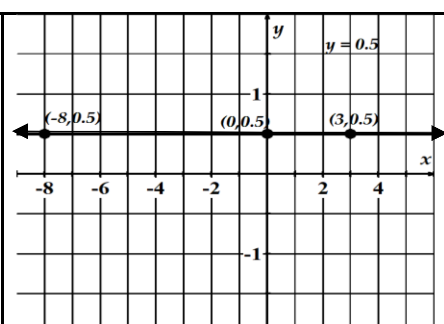
The vertical line crosses the x-axis at $x = -3$, which is the equation of this line passing through (–3, –2).



13)

$y = 0.5$

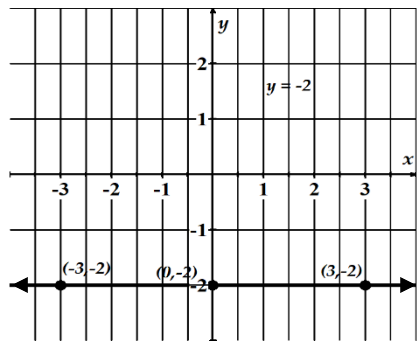
The horizontal line crosses the y-axis at $y = 0.5$, which is the equation of this line passing through (–8, 0.5).



14)

$y = -2$

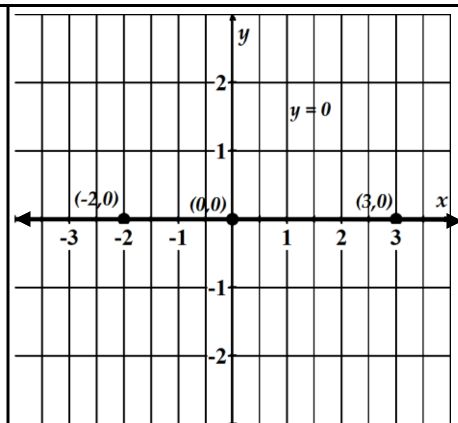
The horizontal line crosses the y-axis at $y = -2$, which is the equation of this line passing through (–3, –2).



15)

$y = 0$

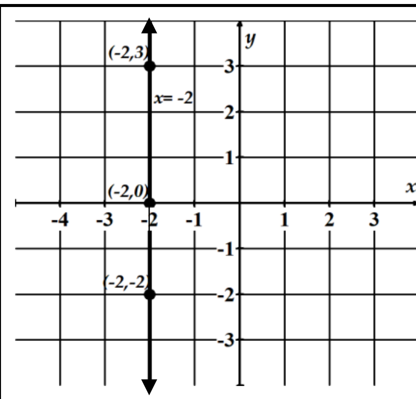
The horizontal line crosses the y-axis at $y = 0$, which is the equation of this line passing through (–2, 0).



16)

$x = -2$

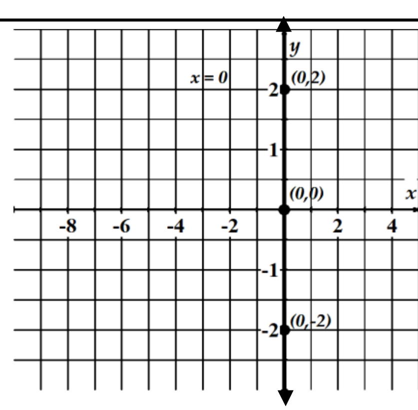
The vertical line crosses the x-axis at $x = -2$, which is the equation of this line passing through (–2, 0).



17)

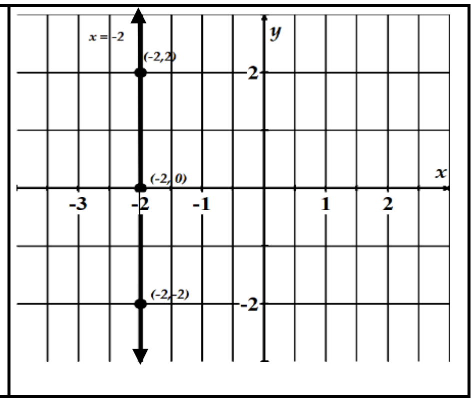
$x = 0$

The vertical line crosses the x-axis at $x = 0$, which is the equation of this line passing through (0, –2).



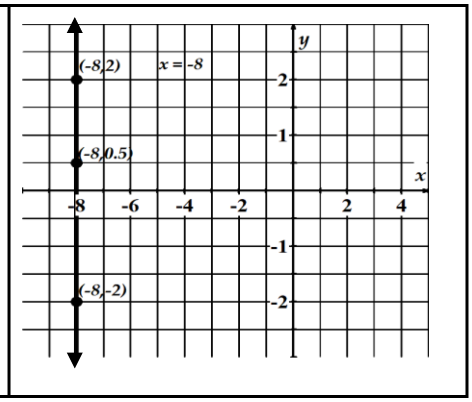
18)

$x = -2$
The vertical line crosses the x-axis at $x = -2$, which is the equation of this line passing through $(-2, -2)$.



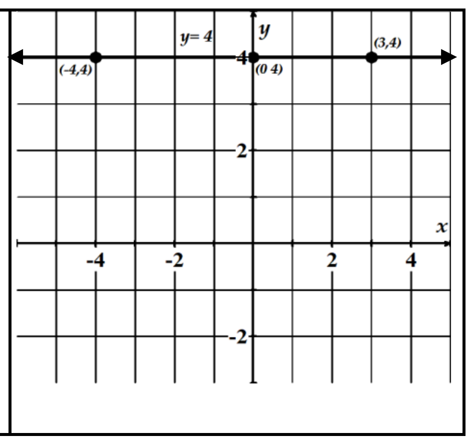
19)

$x = -8$
The vertical line crosses the x-axis at $x = -8$, which is the equation of this line passing through $(-8, 0.5)$.



20)

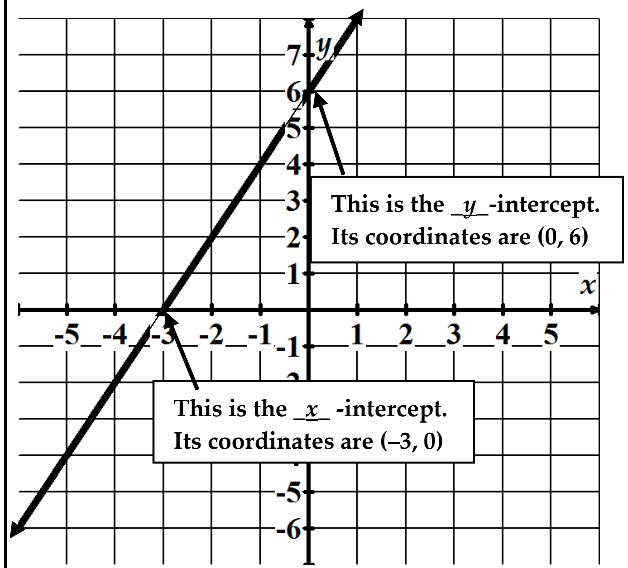
$y = 4$
The horizontal line crosses the y-axis at $y = 4$, which is the equation of this line passing through $(-4, 4)$.



Section 2.7: Summary

1. The **y-intercept (vertical intercept)** of a function is the point where the graph crosses the *y* - axis.	**To find the y-intercept from an equation**, substitute 0 for *x* and solve for *y*.
2. The **x-intercept (horizontal intercept)** of a function is the point where the graph crosses the *x* - axis.	**To find the x-intercept from an equation**, substitute 0 for *y* and solve for *x*.

Example: Find the intercepts from the graph.



Example: Find the intercepts of the equation $3x - y = 9$. Write each as an ordered pair.

y-intercept :

Set $x = 0$.

$3(0) - y = 9$

$-y = 9$ $\boxed{(0, -9)}$

$y = -9$

x-intercept :

Set $y = 0$.

$3x - 0 = 9$

$3x = 9$ $\boxed{(3, 0)}$

$x = 3$

Write the x and y-intercepts of the equation $3x - y = 9$ in the following input-output table:

x	y	(x, y)
0	– 9	(0, – 9)
3	0	(3, 0)

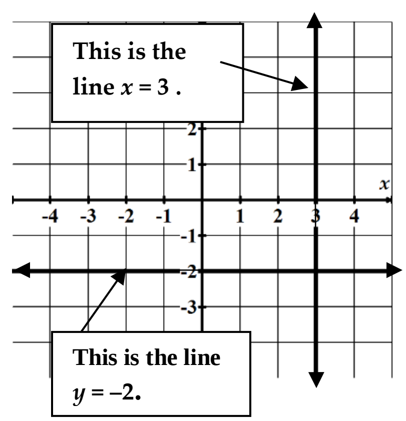
3. For the function $f(x) = -4x + 12$,

a) Finding $f(0) = 12$ gives the __y__-intercept. This is the point (_0_, _12_).

b) Solving $f(x) = 0$ gives the _x_-intercept. This is the point (_3_, _0_).

4. Horizontal Lines

- On a horizontal line all the _y_ - coordinates are the same.
- The equation of a horizontal line through the point (a, b) is ___$y = b$___.
- The equation of the horizontal line that passes through $(3, -2)$ is ___$y = -2$___.



5. Vertical Lines

- On a vertical line all the _x_ - coordinates are the same.
- The equation of a vertical line through the point (a, b) is _$x = a$_.
- The equation of the vertical line thru $(3, -2)$ is ___$x = 3$___.

Section 2.8: Practice Exercises

1) a) The slope of this line is zero, since the line is horizontal.

b) The slope of this line is negative, since from left to right the line is falling.

2) The graph in part b) is steeper

3) (0, 20), (– 4, 0) $slope = \frac{20}{4} = 5$

4) (–3, 12), (–1, 2) $slope = \frac{-10}{2} = -5$

5) $m = 2$, rising

6) no slope, vertical

7) $m = 0$, horizontal

8) $slope = m = \frac{2-1}{-2-3} = \frac{1}{-5} = -\frac{1}{5}$, falling

9) $slope = m = \frac{2-(-3)}{-2-8} = \frac{5}{-10} = -\frac{1}{2}$, falling

10) $m = 0$, horizontal

11) (4, 9.16), (7, 16.03)

$$slope = m = \frac{16.03 - 9.16}{7-4} = \frac{6.87}{3} = 2.29$$

Gasoline cost increases at a rate of $2.29 per gallon.

12) $$slope = m = \frac{189-63}{42-14} = \frac{126}{28} = 4.5$$

Calories increase at a rate of 4.5 calories per one M&M.

Section 2.8: Summary

1. **Slope, *m*,** is a measure of steepness and direction of a line.

2. We will find slope in three ways:

- **Given a table**, find $\frac{\Delta y}{\Delta x}$ for several pairs of rows of the table. If this value is constant, then it is the slope of the line.

Example: Find the slope of the line given by the table.

x	y
-2	7
0	1
3	-8
8	-23

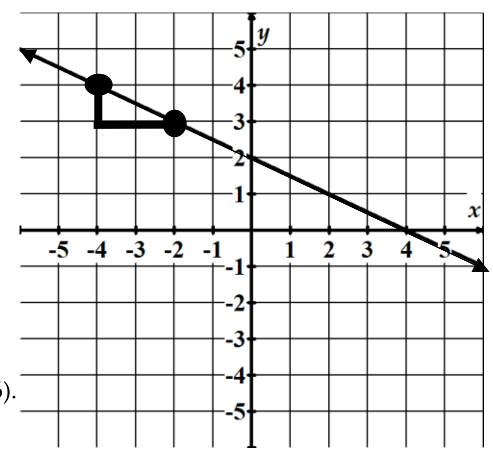
$\Delta x = 0-(-2) = 2$

$\Delta y = 1 - 7 = -6$

$$m = \frac{\Delta y}{\Delta x} = \frac{-6}{2} = -3$$

- **Given a graph**, using two points with integer coordinates, count the $\frac{\Delta y}{\Delta x}$ *or* $\frac{rise}{run}$ to get the slope.

Example: Find the slope of the line graphed to the right.



$$m = -\frac{1}{2}$$

- **Given 2 points** (x_1, y_1) and (x_2, y_2), the slope formula

Is $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$ or $\frac{\text{rise}}{\text{run}}$

Example: Find the slope of the line passing through (2, 3) and (– 4, 6).

$$m = \frac{6-3}{-4-2} = \frac{3}{-6} = -\frac{1}{2}$$

3. The **direction** of a line:

- **Positive** slope means rise (rise or fall) from left to right
- **Negative** slope means fall (rise or fall) from left to right
- **Zero** slope means horizontal (horizontal or vertical)
- **No (undefined)** slope means vertical (horizontal or vertical)

4. **Slope** is also called the **average rate of change** in applications where the input and output variables have units attached to them (miles, dollars, inches, gallons, etc.)

The units for slope are ____ per ____

Units for output — Unit for input

Section 2.9: Practice Exercises

1) $\text{slope} = m = -\frac{1}{10},\ y-\text{intercept} = (0,\ 3)$

$y = -\frac{1}{10}x + 3$

2) $\text{slope} = m = 1,\ y-\text{intercept} = (0,\ 2),\ y = x + 2$

3) $\text{slope} = m = 0,\ y-\text{intercept} = (0,\ 3),\ y = 3$

4) $\text{slope} = m = 10,\ y-\text{intercept} = (0,\ -2)$

$y = 10x - 2$

5) no slope, no $y-$intercept, $x = -20$

6) $\text{slope} = -3,\ y-\text{intercept} = (0,\ 4)$

$y = -3x + 4$

7) $y = -4x - 3$

8) $y = 1$

9) $x = -1$

10) $y = -x + 1$

11) $y = -1$

12) $y = \frac{1}{2}x + 1$

13) $y = \frac{3}{2}x + 2$

14) $y = 5x - 1$

Section 2.9: Summary

1. The equation for a **linear function** can be written as $y = mx + b$ or $f(x) = mx + b$, where

- m is the slope of the line, and
- b is the y-intercept of the line, which is the point (0 , b).

Example: $3x - y = 4$ is a linear equation because it can be written as $y = 3x - 4$.

$m = 3$ is the slope of the line

$b = -4$ gives the y-intercept of the line, which is the point (0, –4)

2. The equation $y = mx + b$ is called **slope-intercept form** because you can quickly find the slope, m, and the y-intercept, b, from this form.

Example: Match the equation with its description.

a)	Line with slope 4 passing through (0, –3)	$y = -3x - 4$	b)
b)	Line with $m = -3$, containing (0. – 4)	$y = -4x + 3$	c)
c)	Line with slope – 4 and y-intercept 3	$y = -3 + 4x$	a)

3. To write a line in slope intercept form, you must know the slope and the y-intercept.

Steps:

I. Find the slope, m.

II. Find b.

III. Write the equation of the line in slope-intercept form, $y = mx + b$.

Example: Use the table below to write an equation of the line.

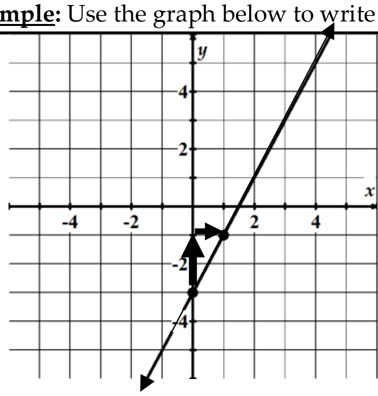
x	y	(x, y)
–1	–12	(–1, –12)
0	–2	(0, –2)
1	8	(1, 8)
2	18	(2, 18)

I. To find the slope, select two points from the table, such as (1, 8) and (2, 18).

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{18-8}{2-1} = \frac{10}{1} = 10$$

II. Find b.	The y-intercept has x = 0. The point in the table that has x = 0 is (0, –2), so b = –2.
III. The equation of the line is	$y = \boxed{10}\,x + \boxed{-2} = 10x - 2$

Example: Use the graph below to write an equation of the line.

	**I.** To find the slope, select two integer points and find $\frac{\text{rise}}{\text{run}}$. We can use (0, – 3) and (1, – 1). To go from (0, – 3) to (1, – 1), we move up 2 spaces and over to the right 1. Therefore the slope is 2.
II. Find b.	The y-intercept is the point where the line crosses the y-axis: (0, – 3), so b = –3.
III. The equation of the line is	$y = \boxed{2}\,x + \boxed{-3} = 2x - 3$

Example: Write the equation of the line with slope m = – 2 that passes through (–3, 5).

I. Find the slope	The slope is given: m = – 2
II. Find b. • Start with $y = mx + b$ • Substitute the given point for x and y and the slope for m. Leave b as the variable. • Solve for b.	Find b. $y = -2x + b$ $5 = -2(-3) + b$ $5 = 6 + b$ $-1 = b$
III. Write the equation of the line in $y = mx + b$ form, by substituting values for m and b, and leaving x and y as variables.	The equation of the line is $y = \boxed{-2}\,x + \boxed{-1} = -2x - 1$

Example: Find the equation of the line that passes through the points (– 3, 5) and (6, 2)

I. Find the slope	The slope of this line is $m = \frac{2-5}{6-(-3)} = -\frac{1}{3}$
II. Find b. • Start with $y = mx + b$ • Substitute a given point for x and y and the slope for m. Leave b as the variable. • Solve for b.	Find b. $y = -\frac{1}{3}x + b$ Use (6, 2) for (x, y): $2 = -\frac{1}{3}(6) + b$ $2 = -2 + b$ $4 = b$

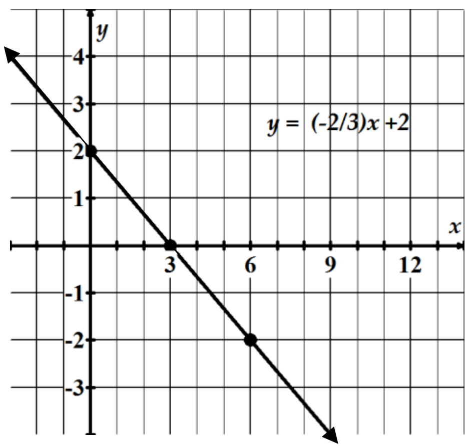
III. Write the equation of the line in $y = mx + b$ form, by substituting values for m and b, leaving x and y as variables.	The equation of the line is $y = \boxed{-\frac{1}{3}}x + \boxed{4} = -\frac{1}{3}x + 4$

4a) Write an equation of a line in which the slope is zero, passing through (0, – 3). $y = -3$.

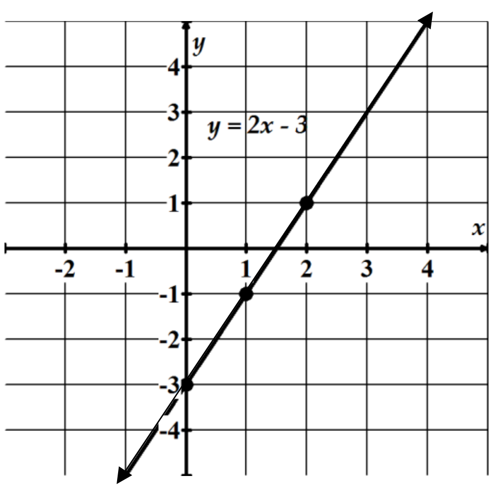
b) Write an equation of a line that has no slope, passing through (0, – 3). $x = 0$.

Section 3.1: Practice Exercises

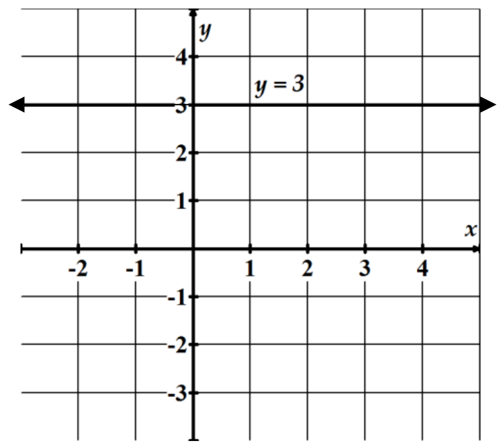
1) $y = -\frac{2}{3}x + 2$



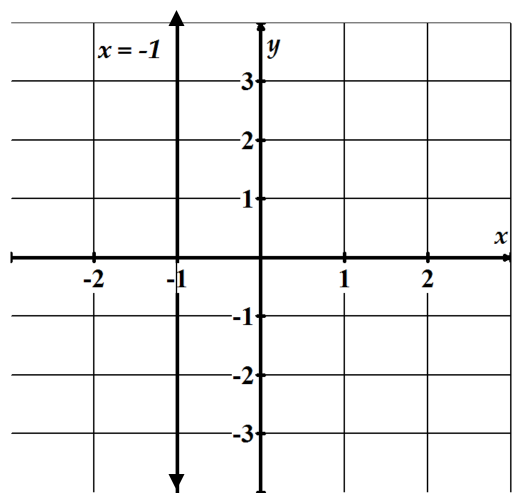
2) $y = 2x - 3$



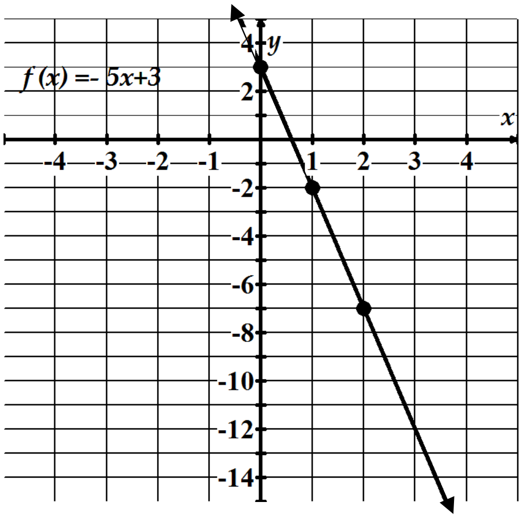
3) $y = 3$



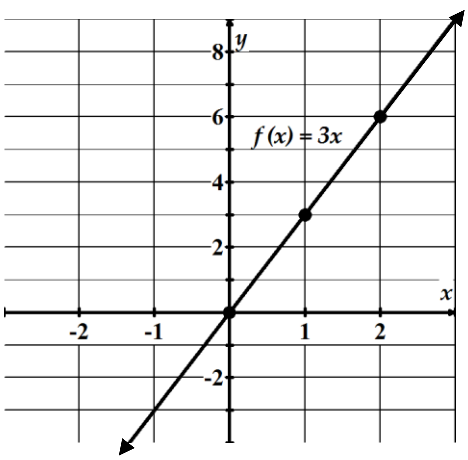
4) $x = -1$



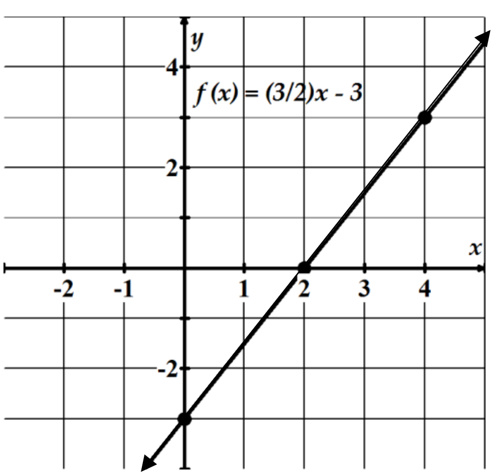
5) $f(x) = -5x + 3$



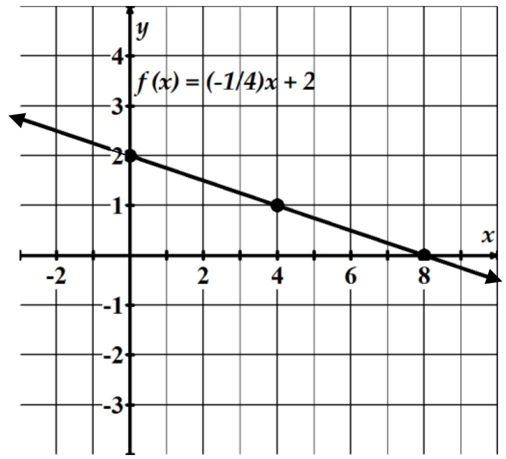
6) $f(x) = 3x$



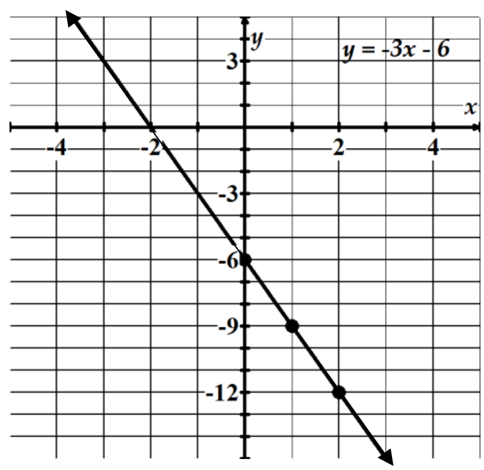
7) $f(x) = \frac{3}{2}x - 3$



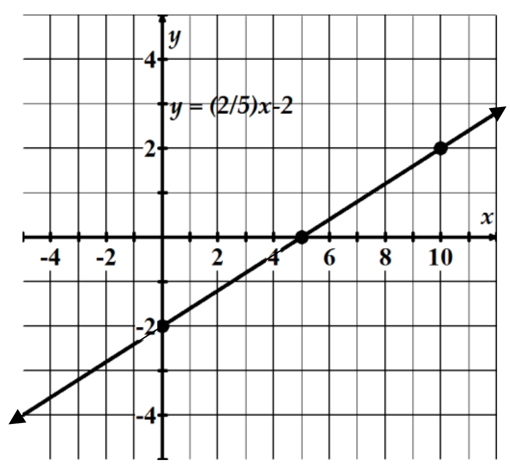
8) $f(x) = -\frac{1}{4}x + 2$



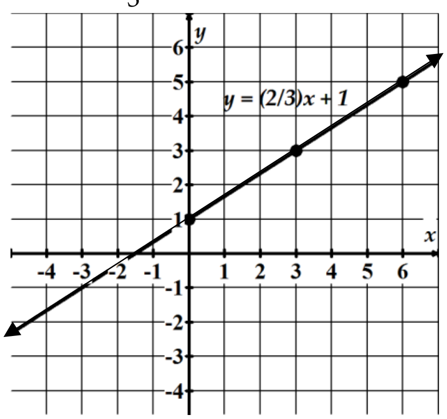
9) $y = -3x - 6$



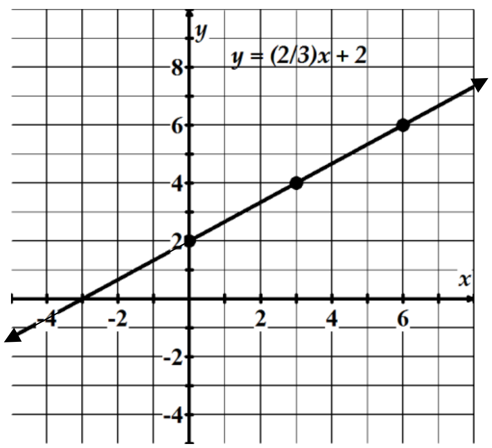
10) $y = \frac{2}{5}x - 2$



11) $y=\frac{2}{3}x+1$



12) $y=\frac{2}{3}x+2$



13) Lines must have same m, such as $y = 4x + 3$

14) Circle the two lines that are parallel:

a) $y = -x - 2$ **c)** $x - y = -2$

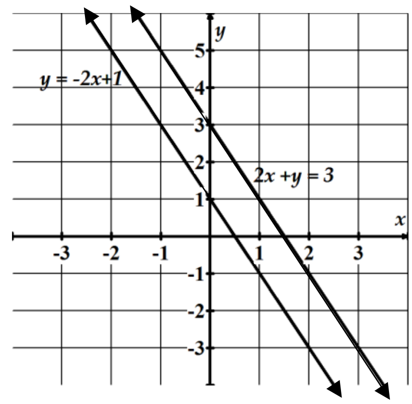
b) $y = -2x - 2$ **d)** $x + y = 8$

15)) $y = -2x + 1$ has slope -2, y-intercept (0, 1)

$2x + y = 3$; $y = -2x + 3$

has slope -2, y-intercept (0, 3).

15) Since these two lines have the same slope and different y-intercepts, these two lines are parallel lines.

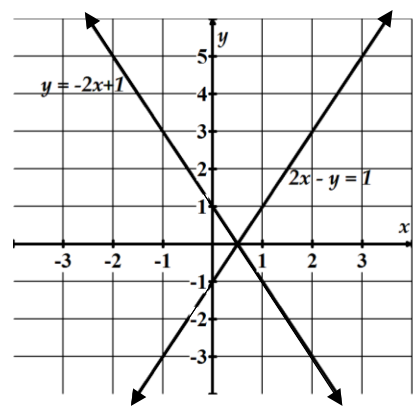


16) $y=-2x+1$ has slope -2

$2x-y=1$

$y=2x-1$ has slope 2

These two slopes are not the same, so these two lines are not parallel.



17) $y=3x+13$

18) $y=2x+10$

19) $y=x+4$

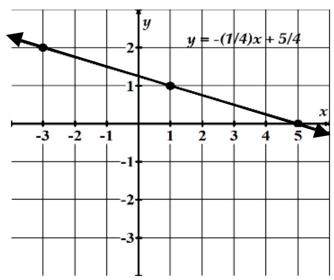
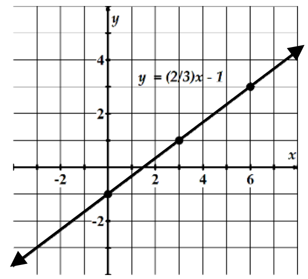
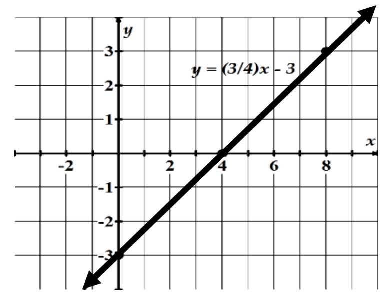
20) $y=4$

Section 3.1: Summary

<u>**Graphing Lines**</u>

1. To graph a line, you need to find at least two points on the line.

If given the slope and a point, ➢ Start at the point and use the slope to find another point. Continue using the slope to get an additional point to help graph a straight line.	If given the equation in slope-intercept form, $y = mx + b$, ➢ Start at the point <u>(0, b)</u> and use the slope to find the next two points.	If given a line in any other form, ➢ Convert to slope-intercept form and proceed as at left. **OR** ➢ Graph by finding the x and y-intercepts.

Graph the line passing through $(-3, 2)$ with slope $m = -\frac{1}{4}$. 	Graph $y = \frac{2}{3}x - 1$. 	Graph $3x - 4y = 12$. 

2. Two lines are $\boxed{\textbf{parallel}}$ if they have the same <u>slope</u> and different <u>*y*-intercept</u>.
<u>Example:</u> $y = -3x + 6$ and $3x + y = -4$ are parallel because they both have slope m = <u>-3</u>.

<u>Example:</u> Find the equation of the line parallel to $x - 5y = 15$ that passes through $(0, 6)$.

I. Find the slope * Solve the given equation for y to find the slope. This will be the slope of the parallel line.	$x - 5y = 15$ Solve for y. $y = \boxed{\frac{1}{5}}x + \boxed{-3} = \frac{1}{5}x - 3$ The slope is $\underline{\frac{1}{5}}$.
II. Find b. * In this example, the y-intercept, b is given.	Write b. <u>Since the line passes through the point (0,6), $b = 6$.</u>
III. Write the equation of the line in $y = mx + b$ form, by substituting values for m and b, leaving x and y as variables.	The equation of the line parallel to the given line is: $y = \boxed{\frac{1}{5}}x + \boxed{6} = \frac{1}{5}x + 6$

Section 3.2: Practice Exercises

1) The slope is \$115 per credit. The cost of tuition for that semester increases at a rate of \$115/credit. The y-intercept is (0, 225). It costs \$225 in fees each semester even if the student hasn't taken any credits (x credits = 0).

2) The slope is \$3.25 per minute. The value remaining on the calling card decreases at a rate of \$3.25/minute. The y-intercept is (0, 25), because when it is new (0 calls have been made), the value of the calling card is \$25.

3) Slope is 8% = 0.08 and it means that Carmine's salary increases at a rate of 0.08 per dollar of sales he made. The y-intercept is (0, 300) which means that his weekly salary is \$300 even when 0 sales are made.
Let x = number of sales he made and y = his weekly salary
The linear function representing this situation is: $f(x) = 0.08x + 300$ or $y = 0.08x + 300$.

4) Slope is -4 and it means that the value remaining on the prepaid card decreases by \$4 per hour. The y-intercept is (0, 40) which means that the beginning value of the prepaid card is \$40 when renting 0 hours of tennis court time.
Let x = number of hours renting a tennis court and y = value remaining on the prepaid card.
The linear function representing this situation is: $f(x) = 40 - 4x$ or $y = 40 - 4x$.

5) **a)** Two ordered pairs: (5, 160), (20, 640). Where x = the amount of time (in minutes) the machine has been operating. y = the number of candy canes produced.

b) $y = 32x$.

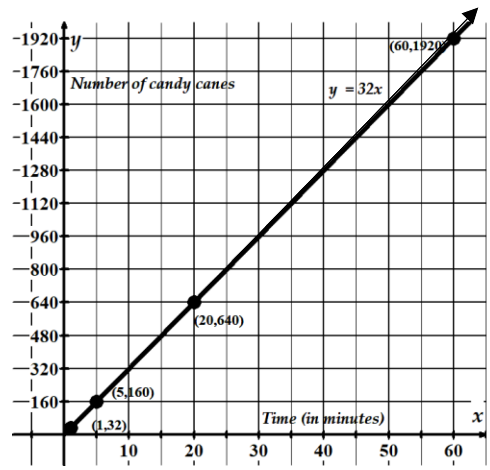
c) The vertical axis intercept is (0, 0) which means at the beginning, $x = 0$ minutes, there are 0 candy canes produced.

d) The horizontal axis intercept is (0, 0) which means at the beginning, $x = 0$, there are no candy canes produced.

e) (1, 32). When machine operates 1 minute, 32 candy canes would be produced.

f) (60, 1920). When the machine operates 1 hour (60 min), 1920 candy canes would be produced.

g)



Section 3.2: Summary

Applications of Linear Functions

1. A linear function can often be used to model applications that contain <u>two</u> quantities.
2. When forming a linear function that models an application, begin by trying to use the <u>Slope-intercept form: $y = mx + b$</u> form of a linear equation.
3. When translating words from an application, the slope is a <u>rate of change</u> which can usually be identified by words such as miles per hour or dollars per credit.
4. To find the vertical axis intercept of a linear function, let <u>x</u> = 0.
5. When forming a linear function from an application, the vertical axis intercept is often thought of as the <u>starting</u> point.
6. **<u>Example</u>**:

A student is depositing \$55 each month into her savings account. If after 6 months her account contains \$830, what was her initial deposit?

a) The input variable, x, represents <u>time in months</u>

b) The output variable, y, represents <u>total money of her account</u>

c) Find the slope and interpret its meaning. <u>Slope is \$55. Deposit \$55 per month</u>

d) To write an equation of the line in $y = mx + b$, replace <u>m</u> with 55.

e) Write the ordered pair that is given in this problem. <u>(6, 830)</u>

f) Find the equation of the line that models this situation:

$y = mx + b$ replace $m = 55$, *the ordered pair* (x, y) is (6, 830) and solve for b

$830 = 55(6) + b$

$830 = 330 + b$

$500 = b$

Therefore the equation of this line is: $y = 55x + 500$.

g) Use the equation of the line to find the initial deposit. If $x = 0$, then the initial deposit is \$500.

h) What is another name for the initial deposit? <u>y-intercept</u>

Section 3.3: Practice Exercises

1)

$3x - y = 5$

$3(2) - (1) \overset{?}{=} 5$

$6 - 1 \overset{?}{=} 5$

$5 = 5 \surd$ true

$x + y = 3$

$(2) + (1) \overset{?}{=} 3$

$3 = 3 \surd$ true

$(2,\ 1)$ is a solution.

2) i.

$2x - y = 1$

$y = 2x - 1$

slope = 2

y – intercept = $(0, -1)$

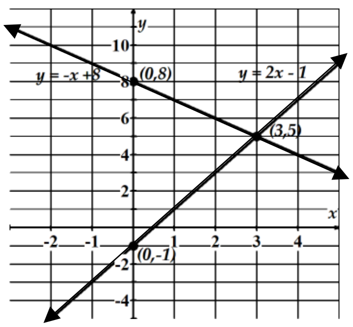
$x + y = 8$

$y = -x + 8$

$slope = -1$

y – intercept = $(0, 8)$

ii.



iii. solution (3, 5)

iv. Check:

$2x - y = 1$

$2(3) - (5) \overset{?}{=} 1$

$6 - 5 \overset{?}{=} 1$

$1 = 1 \surd$ true

$x + y = 8$

$(3) + (5) \overset{?}{=} 8$

$8 = 8 \surd$ true

$(3,\ 5)$ is the solution.

3) i.

$2x + y = 5$

$y = -2x + 5$

$slope = -2$

y – intercept = $(0, 5)$

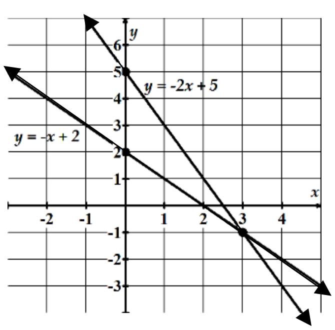
$x + y = 2$

$y = -x + 2$

$slope = -1$

y – intercept = $(0, 2)$

ii.



iii. solution (3, –1)

iv. make sure to check each equation.

4) i.

$-x + y = -1$

$y = x - 1$

$slope = 1$

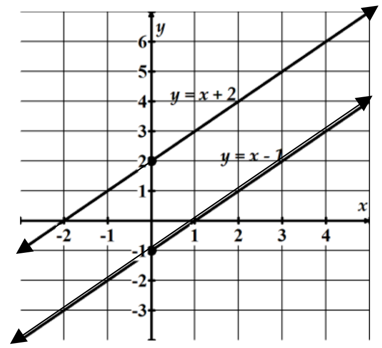
y – intercept = $(0, -1)$

$y = x + 2$

$slope = 1$

y – intercept = $(0, 2)$

ii.



iii. No solutions. Since both lines have the same slope with different y-intercepts, these two lines are parallel.

5) i.

$-2x + 3y = -2$

$3y = 2x - 2$

$y = \frac{2}{3}x - \frac{2}{3}$

$slope = \frac{2}{3}$

y – intercept = $\left(0, -\frac{2}{3}\right)$

$-6x + 9y = -6$

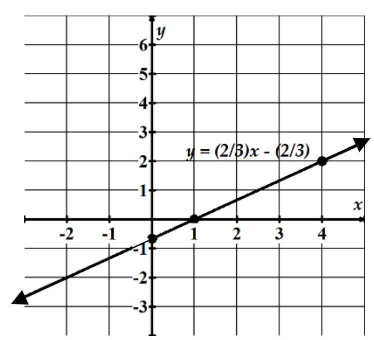
$9y = 6x - 6$

$y = \frac{2}{3}x - \frac{2}{3}$

$slope = \frac{2}{3}$

y – intercept = $\left(0, -\frac{2}{3}\right)$

ii.



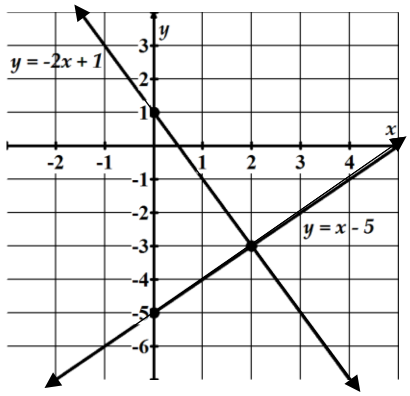
iii. Infinite solutions. Since both lines have the same slope and same y-intercepts, these two lines are coincident.

6) i.

$$2x + y = 1$$
$$y = -2x + 1$$
$$slope = -2$$
$$y - \text{intercept} = (0, 1)$$

$$-x + y = -5$$
$$y = x - 5$$
$$slope = 1$$
$$y - \text{intercept} = (0, -5)$$

ii.



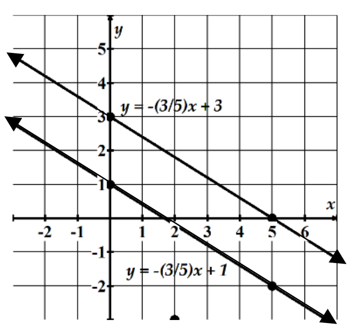
iii. solution (2, –3)
iv. make sure to check each equation.

7) i.

$$3x + 5y = 15$$
$$5y = -3x + 15$$
$$y = -\frac{3}{5}x + 3$$
$$slope = -\frac{3}{5}$$
$$y - \text{intercept} = (0, 3)$$

$$9x + 15y = 15$$
$$15y = -9x + 15$$
$$y = -\frac{3}{5}x + 1$$
$$slope = -\frac{3}{5}$$
$$y - \text{intercept} = (0, 1)$$

ii.



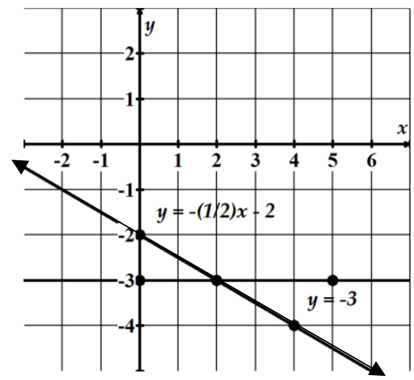
iii. No solutions. Since both lines have the same slope with different y-intercepts, these two lines are parallel.

8) i.

$$y = -3$$
$$slope = 0$$
$$y - \text{intercept} = (0, -3)$$

$$x + 2y = -4$$
$$2y = -x - 4$$
$$y = -\frac{1}{2}x - 2$$
$$slope = -\frac{1}{2}$$
$$y - \text{intercept} = (0, -2)$$

ii.



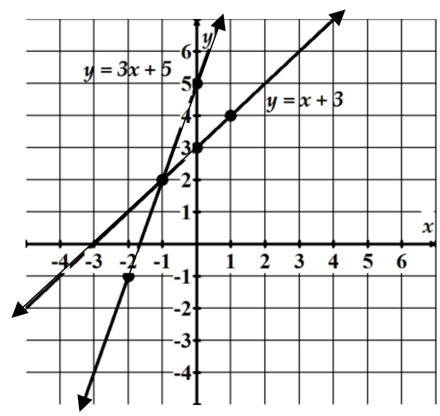
iii. solution (2, –3)
iv. make sure to check each equation.

9) i.

$$-3x + y = 5$$
$$y = 3x + 5$$
$$slope = 3$$
$$y - \text{intercept} = (0, 5)$$

$$-x + y = 3$$
$$y = x + 3$$
$$slope = 1$$
$$y - \text{intercept} = (0, 3)$$

ii.



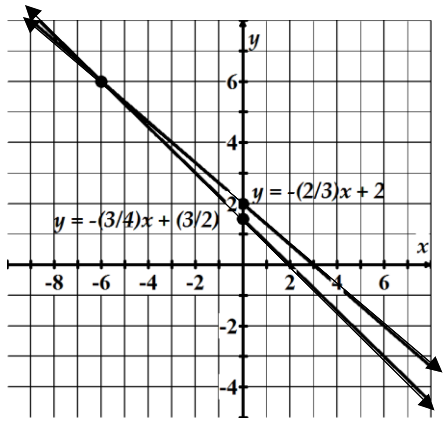
iii. solution (–1, 2)
iv. make sure to check each equation.

10) i.

$$2x + 3y = 6$$
$$3y = -2x + 6$$
$$y = -\frac{2}{3}x + 2$$
$$slope = -\frac{2}{3}$$
$$y - \text{intercept} = (0, 2)$$

$$3x + 4y = 6$$
$$4y = -3x + 6$$
$$y = -\frac{3}{4}x + \frac{3}{2}$$
$$slope = -\frac{3}{4}$$
$$y - \text{intercept} = \left(0, \frac{3}{2}\right)$$

ii.



iii. solution (– 6, 6)

iv. make sure to check each equation.

11) i. $y = 3x - 1$
$y = x - 3$

ii. $\begin{cases} y = 3x - 1 \\ y = x - 3 \end{cases}$

iii. solution $(-1, -4)$

iv. make sure to check each equation.

12) i. $y = x - 8$
$y = -2x + 7$

ii. $\begin{cases} y = x - 8 \\ y = -2x + 7 \end{cases}$

iii. solution (5, –3)

iv. make sure to check each equation.

Section 3.3: Summary

1. To solve a system of linear equations graphically, graph the two <u>lines</u> that represent the equations, and find the point of <u>intersection</u>. This is the solution to the system.
2. A system of linear equations can have <u>0</u>, <u>1</u>, or an <u>infinite</u> number of solutions.

Example

The equation of Line A is
<u>$y = -2x + 4$</u>

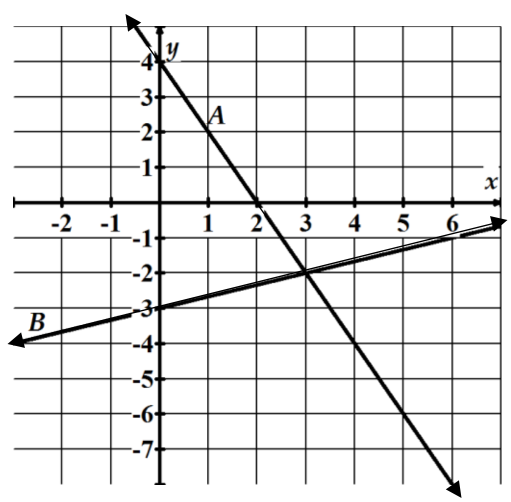
The equation of Line B is <u>$y = \frac{1}{3}x - 3$</u>

The system of equations given at right is:

$$\begin{cases} y = -2x + 4 \\ y = \frac{1}{3}x - 3 \end{cases}$$

The solution to the system is:
<u>$(3, -2)$</u>

Show the checks:



Line A:	Line B:
$y = -2x + 4$ $-2 \stackrel{?}{=} -2(3) + 4$ $-2 \stackrel{?}{=} -6 + 4$ $-2 = -2 \surd$	$y = \frac{1}{3}x - 3$ $-2 \stackrel{?}{=} \frac{1}{3}(3) - 3$ $-2 \stackrel{?}{=} 1 - 3$ $-2 = -2 \surd$

Section 3.4: Practice Exercises

1)

$x=-1$
$3x-2y=10$
$3(-1)-2y=10$
$-3-2y=10$
$-2y=13$
$y=-\frac{13}{2}$ the solution is $\left(-1,-\frac{13}{2}\right)$

Be sure to check the solution.

2)

$y=5$
$2x+3y=8$
$2x+3(5)=8$
$2x+15=8$
$2x=-7$
$x=-3.5$ the solution is $(-3.5, 5)$

Be sure to check the solution.

3)

$y=3x-2$
$2x-3(3x-2)=-1$
$2x-9x+6=-1$
$-7x=-7$
$x=1$

$y=3x-2$
$y=3(1)-2$
$y=1$
the solution is $(1,1)$
Be sure to check the solution.

4)

$x=y-2$
$2x-3y=7$
$2(y-2)-3y=7$
$2y-4-3y=7$
$-y=11$
$y=-11$

$x=y-2$
$x=-11-2$
$x=-13$
the solution is $(-13, -11)$
Be sure to check the solution.

5)

$y=2$
$2x-3y=8$
$2x-3(2)=8$
$2x-6=8$
$2x=14$
$x=7$ The solution is $(7, 2)$

Be sure to check the solution.

6)

$x=-1$
$3x-3y=6$
$3(-1)-3y=6$
$-3-3y=6$
$-3y=9$
$y=-3$ The solution is $(-1, -3)$

Be sure to check the solution

7)

$y=2x$
$x+3y=14$
$x+3(2x)=14$
$x+6x=14$
$7x=14$
$x=2$

$y=2x$
$y=2(2)=4$
the solution is $(2, 4)$
Be sure to check the solution.

8)

$3x+y=7$
$y=-3x+7$

$2x+3y=14$
$2x+3(-3x+7)=14$
$2x-9x+21=14$
$-7x=-7$
$x=1$
$y=-3x+7$
$y=-3(1)+7$
$y=4$
the solution is $(1, 4)$
Be sure to check the solution.

9)

$4x+y=7$
$y=-4x+7$

$5x-8y=18$
$5x-8(-4x+7)=18$
$5x+32x-56=18$
$37x=74$
$x=2$
$y=-4x+7$
$y=-4(2)+7$
$y=-1$
the solution is $(2, -1)$
Be sure to check the solution.

10)

$2x-y=1$
$y=2x-1$

$y=2x-1$
$4x-2y=4$
$4x-2(2x-1)=4$
$4x-4x+2=4$
$2=4$ *f*alse
No solutions.
These two lines are parallel.

11)

$x+2y=2$
$x=-2y+2$

$4x+8y=8$
$4(-2y+2)+8y=8$
$-8y+8+8y=8$
$8=8$ true
Infinte solutions.
Since both lines have the same slope
and same $y-$intercepts,
these two lines are coincident.

12)

$x = 4y - 14$
$3x + 5y = 9$
$3(4y - 14) + 5y = 9$
$12y - 42 + 5y = 9$
$17y = 51$
$y = 3$

$x = 4y - 14$
$x = 4(3) - 14$
$x = -2$
the solution is $(-2,\ 3)$
Be sure to check the solution.

14)

$8 - 3y = 3y - 4$
$-6y = -12$
$y = 2$
$x = 8 - 3y$
$x = 8 - 3(2)$
$x = 2$
the solution is $(2,\ 2)$
Be sure to check the solution.

13)

$3x - 1 = 2x + 4$
$x = 5$
$y = 3x - 1$
$y = 3(5) - 1 = 14$
the solution is $(5,\ 14)$
Be sure to check the solution.

Section 3.4: Summary

WORK		STEP
$\begin{cases} 2x - 3y = 2 \\ x = 2y - 4 \end{cases}$		**1. Solve one equation for a variable.** **For this system, it will be easiest to solve the second equation for *x*. To get *x* to stand alone on one side, simply add <u>2*y*</u> to both sides of the equation.**
$2(2y - 4) - 3y = 2$		**2. Substitute the expression you got in step 1 for the *x* in the untouched (first) equation. The expression should be placed in parentheses, as distribution will most likely be used.**
$4y - 8 - 3y = 2$ $y - 8 = 2$ $y = 10$		**3. The result is that the first equation now has one variable. Solve this equation for *y*.**
$x = 2(10) - 4$ $x = 16$		**4. Substitute the solution for *y* into one of the original equations, and solve for *x*.**
$x = 16,\ y = 10$ or $(16, 10)$		**5. Write the solution either as an ordered pair ordered pair (*x*, *y*), or in "*x* = ___, *y* = ___" form.**
$2x - 3y = 2$ $2(16) - 3(10) \stackrel{?}{=} 2$ $32 - 30 = 2$ *true*	$x - 2y = -4$ $16 - 2(10) \stackrel{?}{=} -4$ $16 - 20 = -4$ *true*	**6. Check the solution in both of the original equations.**
The solution checks in both equations.		

Interpreting Results From the Substitution Method

- There is ***one solution*** (x, y) to the system when a value can determined for both x and y.
- There are ***infinitely many solutions*** to the system when the variables are gone and a true statement remains (for example, 5 = 5).
- There are ***no solutions*** to the system when the variables are gone and a false statement remains (for example, 2 = 4).

Example: $\begin{cases} -8x-2y=1 \\ 4x+y=2 \end{cases}$

Solution:

$$4x+y=2$$
$$y=2-4x$$

$$-8x-2y=1$$
$$-8x-2(2-4x)=1$$
$$-8x-4+8x=1$$
$$-4\neq 1 \quad \textit{false}$$

therefore the system has no solutions.

These two lines are parellel

Section 3.5: Practice Exercises

1)

$$\begin{aligned} 4x-3y&=-7 \\ \underline{5x+3y}&\underline{=25} \\ 9x&=18 \\ x&=2 \end{aligned}$$

$$\begin{aligned} 4x-3y&=-7 \\ 4(2)-3y&=-7 \\ 8-3y&=-7 \\ -3y&=-15 \\ y&=5 \end{aligned}$$

The solution is $(2,5)$

Be sure to check the solution.

2)

$$\begin{aligned} 2x-3y&=-7 \\ \underline{-2x-3y}&\underline{=25} \\ -6y&=18 \\ y&=-3 \end{aligned}$$

$$\begin{aligned} 2x-3y&=-7 \\ 2x-3(-3)&=-7 \\ 2x+9&=-7 \\ 2x&=-16 \\ x&=-8 \end{aligned}$$

The solution is $(-8,-3)$

Be sure to check the solution.

3)

$$\begin{aligned} 3x+y&=1 \\ -1(5x+y=3)\rightarrow \underline{-5x-y}&\underline{=-3} \\ -2x&=-2 \\ x&=1 \end{aligned}$$

$$\begin{aligned} 3x+y&=1 \\ 3(1)+y&=1 \\ 3+y&=1 \\ y&=-2 \end{aligned}$$

The solution is $(1,-2)$

Be sure to check the solution

4)

$$\begin{aligned} x+4y&=1 \\ -1(x-2y=-5)\rightarrow \underline{-x+2y}&\underline{=5} \\ 6y&=6 \\ y&=1 \end{aligned}$$

The solution is $(-3,1)$

$$x+4y=1$$
$$x+4(1)=1;\ x=-3$$

Be sure to check the solution

5)

$$\begin{aligned} 2x+3y&=-10 \\ 2(-x+2y=-2)\rightarrow \underline{-2x+4y}&\underline{=-4} \\ 7y&=-14 \\ y&=-2 \end{aligned}$$

$$\begin{aligned} -x+2y&=-2 \\ -x+2(-2)&=-2 \\ -x-4&=-2 \\ -x&=2 \\ x&=-2 \end{aligned}$$

The solution is $(-2,-2)$

Be sure to check the solution

6)

$$\begin{aligned} 8x-6y&=4 \\ -2(5x-3y=1)\rightarrow \underline{-10x+6y}&\underline{=-2} \\ -2x&=2 \\ x&=-1 \end{aligned}$$

$$\begin{aligned} 5x-3y&=1 \\ 5(-1)-3y&=1 \\ -5-3y&=1 \\ -3y&=6 \\ y&=-2 \end{aligned}$$

The solution is $(-1,-2)$

Be sure to check the solution

7)

$$\begin{aligned} 8(3x-5y=9)&\rightarrow 24x-40y=72 \\ 5(4x+8y=12)&\rightarrow \underline{20x+40y=60} \\ &\quad 44x=132 \\ &\quad x=3 \end{aligned}$$

$$\begin{aligned} 3x-5y&=9 \\ 3(3)-5y&=9 \\ 9-5y&=9 \\ -5y&=0 \\ y&=0 \end{aligned}$$

The solution is $(3,0)$

Be sure to check the solution

8)

$$\begin{aligned} 2(5x-2y=-13) &\rightarrow \quad 10x-4y=-26 \\ -5(2x-3y=-14) &\rightarrow \underline{-10x+15y=70} \\ &\qquad\qquad 11y=44 \\ &\qquad\qquad y=4 \end{aligned}$$

$5x-2y=-13$

$5x-2(4)=-13$

$5x-8=-13$

$5x=-5$

$x=-1$

The solution is $(-1,4)$

Be sure to check the solution

9)

$$\begin{aligned} 5(3x-6y=0) &\rightarrow \quad 15x-30y=0 \\ -3(5x-4y=-6) &\rightarrow \underline{-15x+12y=18} \\ &\qquad\qquad -18y=18 \\ &\qquad\qquad y=-1 \end{aligned}$$

$3x-6y=0$

$3x-6(-1)=0$

$3x+6=0$

$3x=-6$

$x=-2$

The solution is $(-2,-1)$

Be sure to check the solution

10)

$$\begin{aligned} 3(2x-5y=19) &\rightarrow \ 6x-15y=57 \\ 5(5x+3y=1) &\rightarrow \ \underline{25x+15y=5} \\ &\qquad 31x=62 \\ &\qquad x=2 \end{aligned}$$

$5x+3y=1$

$5(2)+3y=1$

$10+3y=1$

$3y=-9$

$y=-3$

The solution is $(2,-3)$

Be sure to check the solution

11)

$$\begin{aligned} x+6y&=8 \\ \underline{-x-2y}&\underline{=0} \\ 4y&=8 \\ y&=2 \end{aligned}$$

$x+6y=8$

$x+6(2)=8$

$x+12=8$

$x=-4$

The solution is $(-4,2)$

Be sure to check the solution

12)

$$\begin{aligned} & \qquad\qquad 6x-5y=14 \\ -3(2x+2y=-10) &\rightarrow \underline{-6x-6y=30} \\ &\qquad\qquad -11y=44 \\ &\qquad\qquad y=-4 \end{aligned}$$

$2x+2y=-10$

$2x+2(-4)=-10$

$2x+-8=-10$

$2x=-2$

$x=-1$

The solution is $(-1,-4)$

Be sure to check the solution

13)

$$\begin{aligned} 3(4x+8y=8) &\rightarrow \ 12x+24y=24 \\ -4(3x+6y=6) &\rightarrow \underline{-12x-24y=-24} \\ &\qquad\qquad 0=0 \quad \textit{true} \end{aligned}$$

Infinite solutions

These two lines are coincident.

14)

$$\begin{aligned} 2x+y&=-1 \\ \underline{3x-y}&\underline{=-9} \end{aligned}$$

$5x=-10$

$x=-2$

$2x+y=-1$

$2(-2)+y=-1$

$-4+y=-1$

$y=3$

The solution is $(-2,3)$

Be sure to check the solution

15)

$$\begin{aligned} & \qquad\qquad 3x+4y=25 \\ -1(3x-3y=4) &\rightarrow \underline{-3x+3y=-4} \\ &\qquad\qquad 7y=21 \\ &\qquad\qquad y=3 \end{aligned}$$

$3x+4y=25$

$3x+4(3)=25$

$3x+12=25$

$3x=13, \quad x=\frac{13}{3}$

The solution is $\left(\frac{13}{3},3\right)$

Be sure to check the solution

16)

$$-2(2x-y=1)\rightarrow -4x+2y=-2$$
$$\underline{4x-2y=4}$$
$$0=2 \quad \textit{false}$$

No solutions
These two lines are parallel.

17)

$$2(x-2y=-2)\rightarrow 2x-4y=-4$$
$$\underline{-2x+4y=4}$$
$$0=0 \quad \textit{true}$$

Infinite solutions
These two lines are coincident.

18)

$$5(5x+2y=8)\rightarrow 25x+10y=40$$
$$2(3x-5y=11)\rightarrow \underline{6x-10y=22}$$
$$31x=62; \quad x=2$$
$$5x+2y=8$$
$$5(2)+2y=8$$
$$10+2y=8; \quad 2y=-2; \quad y=-1$$
$$\textit{The solution is } (2,-1)$$

Be sure to check the solution

19)

$$4(4x+3y=1)\rightarrow 16x+12y=4$$
$$3(5x-4y=9)\rightarrow \underline{15x-12y=27}$$
$$31x=31; \quad x=1$$
$$4x+3y=1$$
$$4(1)+3y=1$$
$$3y=-3; \quad y=-1$$
$$\textit{The solution is } (1,-1)$$

Be sure to check the solution

20)

$$5(3x+2y=-4)\rightarrow 15x+10y=-20$$
$$2(4x-5y=10)\rightarrow \underline{8x-10y=20}$$
$$23x=0; \quad x=0$$
$$3x+2y=-4$$
$$3(0)+2y=-4$$
$$2y=-4; \quad y=-2$$
$$\textit{The solution is } (0,-2)$$

Be sure to check the solution

Section 3.5: Summary

Example: $\begin{cases} 2x-3y=7 \\ -6x+2y=21 \end{cases}$

WORK	STEP
$2x-3y=7$ $-6x+2y=21$	**1. Make sure to write the equations so that the like terms line up vertically.**
$3(2x-3y=7)\rightarrow \quad 6x-9y=21$ $-6x+2y=21 \quad \rightarrow \underline{-6x+2y=21}$	**2. Multiply one or both equations to create opposite coefficients on one variable.**
$-7y=42$ $y=-6$	**3. Add the resulting equations by combining like terms. This should eliminate one variable. Solve for the remaining variable.**
$-6x+2y=21$ $-6x+2(-6)=21$ $-6x-12=21$ $-6x=33$ $x=-\frac{33}{6}=-\frac{11}{2}$ $\textit{The solution is } \left(-\frac{11}{2},-6\right)$	**3. Substitute the solution from step 3 into either of the two original equations, and solve for the other variable. State the solution as an ordered pair (*x*, *y*) or in "*x* = ___, *y* = ___" form.**
$-6x+2y=21$ $-6\left(-\frac{11}{2}\right)+2(-6)=21\surd$ $2x-3y=7$ $2\left(-\frac{11}{2}\right)-3(-6)=7\surd$	**5. Check the solution in both of the original equations.**

Example: $\begin{cases} 5x - 15y = 45 \\ -x + 3y = -9 \end{cases}$

$$\begin{aligned} &5(-x+3y=-9) \\ &-5x+15y=-45 \\ &\underline{5x-15y=45} \\ &0=0 \quad \textit{true} \end{aligned}$$

Infinite solutions

These two lines are coincident

Applications:

Example: The perimeter of a rectangle is 76 inches. Its length is two inches less than three times its width. What are the dimensions of the rectangle?

In this case, we can draw a diagram to illustrate:

length is 2 less than 3 times width

Perimeter is 76 inches

width

Work	Step
Let L = the length of this rectangle W = the width of this rectangle $P = 2\,L + 2\,W$ where P is perimeter	**1. Define variables to represent the length and width. Write an equation expressing the perimeter of the rectangle in terms of the length and the width.**
$\begin{cases} 2L + 2W = 76 \\ L = 3W - 2 \end{cases}$	**2. Write the system of linear equations expressing the length and the width.**
$2L + 2W = 76$ $2(3W - 2) + 2W = 76$ $6W - 4 + 2W = 76$ $8W = 80$ $W = 10$	**3. Solve this system of linear equations from step 2 by substitution. Solve for the width.**
$L = 3W - 2$ $L = 3(10) - 2$ $L = 30 - 2 = 28$	**4. Substitute the value into either of the original equations. Solve for the length.**
The length of the rectangle is 28 inches and the width is 10 inches.	**5. State your solution clearly.**
2(28) + 2(10) = 76 which is the perimeter 28 = 3(10) – 2 true	**6. Check the solution in your two equations.**

Applications:

Example:

At a back-to-school sale, Lydia bought 3 pens and 5 notebooks and paid $11.42 before tax. Carolina bought 4 of the same pens and 2 of the same notebooks and paid $5.94 before tax. How much did each pen cost? How much did each notebook cost?

In this example, a system of equations is appropriate. First, define the two variables:

Let x = cost of one pen

Let y = cost of one notebook

Each of the first two sentences contains three numbers.

"… Lydia bought 3 pens and 5 notebooks and paid $11.42…"

"Carolina bought 4 of the same pens and 2 of the same notebooks and paid $5.94…"

Those three numbers become A, B, and C in our standard $Ax + By = C$ equations. Thus, our system of equations is:

$$\begin{cases} 3x+5y=11.42 \\ 4x+2y=5.94 \end{cases}$$

Solve this system of equations by elimination.

$$\begin{aligned} 2(3x+5y=11.42) &\rightarrow \quad 6x+10y=22.84 \\ -5(4x+2y=5.94) &\rightarrow \underline{-20x-10y=-29.70} \\ & \qquad -14x=-6.86 \\ & \qquad\quad x=0.49 \end{aligned}$$

$$\begin{aligned} &3x+5y=11.42 \\ &3(0.49)+5y=11.42 \\ &1.47+5y=11.42 \\ &5y=9.95 \\ &y=1.99 \end{aligned}$$

Answer:
Each pen costs __$0.49________.
Each notebook costs __$1.99________.

Check: Be sure to check both equations

Section 3.6: Practice Exercises

1) A professor asked the twenty students in her Statistics class to write down the average number of hours they sleep each night, to the nearest half hour. The results are in the table below.

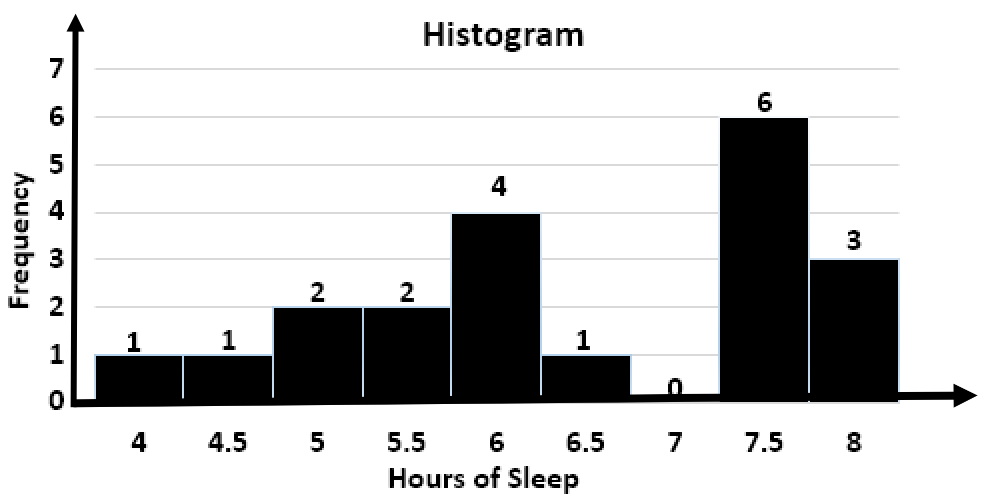
Student	Hours of Sleep
Lisa	4
Carolina	7.5
Lydia	5
Raemin	7.5
Mian	6
Jay	6.5
Bobby	6
Clay	8
Jade	4.5
Reo	5.5

Student	Hours of Sleep
Mariama	6
Deslyn	5
Celina	5.5
Daniella	8
Stephanie	7.5
Elise	7.5
Paolo	6
Emma	8
Brooke	7.5
Angelica	7.5

a) Name the quantitative variable.________________________hours of sleep____________.

b) Create a frequency distribution table and histogram for the hours of sleep.

Quantitative variable: Hours of Sleep	Tally	Frequency (f)	Relative frequency (rf)
4	/	1	1/20 = 0.05
4.5	/	1	1/20 = 0.05
5	//	2	2/20 = 0.10
5.5	//	2	2/20 = 0.10
6	////	4	4/20 = 0.20
6.5	/	1	1/20 = 0.05
7		0	0
7.5	~~////~~ /	6	6/20 = 0.30
8	///	3	3/20 = 0.15
		$\sum f = 20$	$\sum rf = 1$



c) Find the mean, median and mode for the hours of sleep of the Statistics students.

$\sum x = 1(4)+1(4.5)+2(5)+2(5.5)+4(6)+1(6.5)+6(7.5)+3(8)=129$

The <u>mean</u> is $\bar{x} = \frac{\sum x}{n} = \frac{129}{20} = 6.45.$

The average hours of sleep for the selected students is 6.45 hours.

To find the median, we have to order from least to greatest, we have 4, 4.5, 5, 5, 5.5, 5.5, 6, 6, 6, 6, 6.5, 7.5, 7.5, 7.5, 7.5, 7.5, 7.5, 8, 8, 8. Since there are 20 students which is an even number, the median is the mean of the two middle data entries. For the 20 students, the middle will be $\frac{20}{2} = 10th\ place.$

The <u>median</u> is $\frac{10^{th}\text{ place} + 11^{th}\text{ place}}{2} = \frac{6+6.5}{2} = \frac{12.5}{2} = 6.25$ hours.

The <u>mode</u> is the number of hours of sleep that occurs the most which is 7.5 hours

d) Relative frequency for 6 hours of sleep is $\frac{4}{20} \approx 0.20$

2) A random sample of the income of ten professional athletes produced a set of data, where x is the number of commercial endorsements the player has and y is the amount of money made (in millions of dollars). The regression line that relates this data is: $y = 2.23 + 1.00x$.

a) What is the slope of this line? Give an interpretation of the slope.
$m = 1$. There is an increase of $1million made for every number of commercial endorsements the player has.

b) What is the y-intercept? Explain what the y-intercept means for this data.
Since $b = 2.23$, the y-intercept is (0, 2.23). When there are no commercial endorsements, the amount of money made is $2.23 millions of dollars.

c) If a professional athlete has 7 endorsements, how much money will she be expected to make?
$y = 2.23 + 1(7) = 9.23$
When there are 7 endorsements, she will expect to make $9.23 million.

d) How many endorsements should a professional athlete have if she wants to have an income of approximately 5 million dollars?
$5 = 2.23 + 1.00x \rightarrow 2.77 = x$
She will need approximate 2.77 endorsements if she wants to have an income of $5 million dollars.

Section 3.6: Summary

1) Classify the following data as qualitative or quantitative.

a) Name ______qualitative______
b) Age ______quantitative______
c) Height ______quantitative______
d) Eye Color ______qualitative______
e) Phone number ______qualitative______

2) A ______bar graph______ is a graph to display qualitative variables.

3) A ______histogram______ is a graph to display quantitative variables.

4) State True/False for the following statements.

a) Mean of a data set always exists. True

b) Mode of a data set always exists. False

c) An outlier (a really high or really low data value) has no effect on mean of the data. False

d) To find median, the data must be arranged in an increasing or decreasing order. True

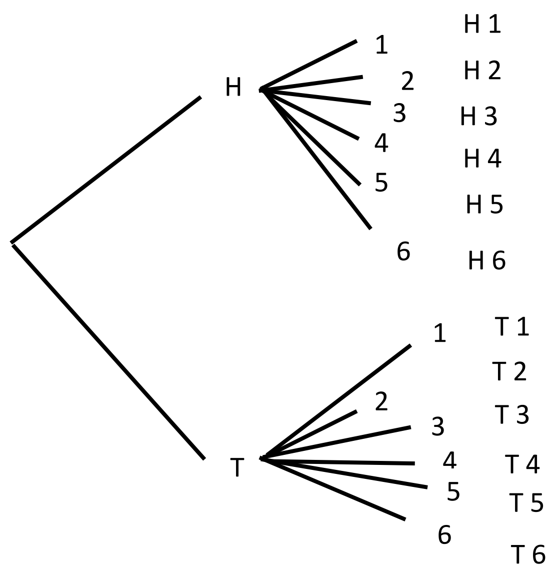
5) For an auto insurance company the premium is given by the following equation:
$y = 250 - 10x$, where x is the number of years of accident-free driving and y is the quarterly premium.

a) What is the slope for this equation? Interpret the slope for this situation.
$m = -10$. The price of the premium goes down $10 for each year of accident-free driving.

b) What is the y-intercept for this equation? Interpret the y-intercept for this situation.
$b = 250$. The price of the quarterly premium is $250 for a new customer.

Section 3.7: Practice Exercises

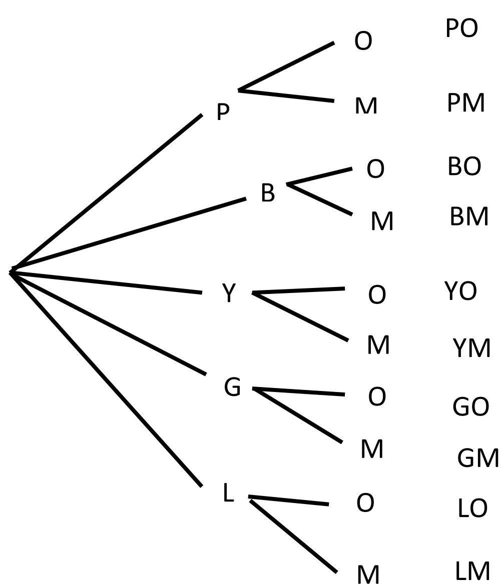
1) a)



b) There are 12 different outcomes. {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}.

c) 2 (6) = 12. There are 12 different outcomes

2) a)



b) Carolina has 10 different outfits. {PO, PM, BO, BM, YO, YM, GO, GM, LO, LM}

c) 5 (2) = 10 different outfits.

3) **a)** The theoretical probability of one coin toss is heads is $\frac{1}{2}$ or 0.5.

b)

Heads	**Tails**
The probability of getting a head if flipping a coin 10 times is $\frac{7}{10} = 0.7$	The probability of getting a tail if flipping a coin 10 times is $\frac{3}{10} = 0.3$
The probability of getting a head if flipping a coin 100 times is $\frac{56}{100} = 0.56$	The probability of getting a tail if flipping a coin 100 times is $\frac{44}{100} = 0.44$
The probability of getting a head if flipping a coin 1000 times is $\frac{492}{1000} = 0.492$	The probability of getting a head if flipping a coin 1000 times is $\frac{508}{1000} = 0.508$

As the number of coin flips increases, the probability of getting a head gets closer to the theoretical probability, which is $\frac{1}{2}$ or 0.5. Similarly, if you flip a coin 1000 times, the probability of getting a tail is also close to $\frac{1}{2}$ or 0.5. As the number of flips increases, the empirical probability gets closer to the theoretical probability.

Section 3.7: Summary

1) A twelve-sided die (with sides that are numbered from 1 to 12) is rolled once. Find the probability that it lands on a number that is:

a) greater than 6 $\frac{6}{12} = \frac{1}{2}$

b) a multiple of 5 $\frac{2}{12} = \frac{1}{6}$

c) an even number $\frac{6}{12} = \frac{1}{2}$

d) at most 7 $\frac{7}{12}$

2) A <u>tree diagram</u> can be used to list the sample space.

3) The <u>Fundamental Counting</u> principle can be used to calculate size of a large sample space without listing all possible outcomes. This principle states that if there are m ways to do one step of an experiment and n ways to do the second step, there are $m \cdot n$ ways to do both steps

4) State True/False for the following statements and **EXPLAIN** briefly why it is true or false.

a) Probability can be greater than 1.
<u>False</u>, because a probability of 1 means that the event is certain to happen.

b) As number of trials of an experiment increases, the empirical probability increases.
<u>False</u>. As the number of trials of an experiment increases, the empirical probability gets closer in value to the theoretical probability.

c) If there are 4 different shirts and 5 different pants to choose from to make an outfit, there are 20 different possible outfits consisting of a shirt and pants.
<u>True</u>, based on the Fundamental Counting principle.

d) If there are 12 red balls and 16 green balls in an urn, the probability of picking a red ball is $\frac{12}{16}$.

False. The probability of picking a red ball is 12 divided by the total number of balls (12+16=28). So the probability of picking a red ball is $\frac{12}{28}=\frac{3}{7}$.

Section 3.8: Practice Exercises

1) Let x = # of calories in each peanut
y = # of calories in each cashew

$$\begin{cases} 16x+5y=135 \\ 20x+25y=405 \end{cases}$$

$$\begin{aligned} -5(16x+5y=135) \rightarrow -80x-25y&=-675 \\ 20x+25y&=405 \\ \hline -60x&=-270 \\ x&=4.5 \end{aligned}$$

$16x+5y=135$
$16(4.5)+5y=135$
$72+5y=135$
$5y=63$
$y=12.6$ each peanut has 4.5 calories and each cashew has 12.6 calories

2) Let x = Price of an adult ticket
y = Price of a student ticket

$$\begin{cases} 12x+10y=160 \\ 8x+15y=152.50 \end{cases}$$

$$\begin{aligned} 3(12x+10y=160) \rightarrow \quad 36x+30y&=480 \\ -2(8x+15y=152.50) \rightarrow -16x-30y&=-305 \\ \hline 20x&=175 \\ x&=8.75 \end{aligned}$$

$12x+10y=160$
$12(8.75)+10y=160$
$105+10y=160$
$10y=55$
$y=5.5$ An adult ticket costs \$8.75 and a student ticket costs \$5.50

3) **a)** tuition at the community college
$y=450+135x$

b) tuition at vocational school
$y=300+150x$

c) $450+135x=300+150x$
$x=10$
At 10 credits, the tuition is the same.

d) Community College:
$450+135x=450+135(12)=2070$
Vocational School:
$300+150x=300+150(12)=2100$
community college tuition is cheaper

4) **a)** cost of renting a car at Axis

$y = 135 + 0.35x$

cost of renting a car at Histz

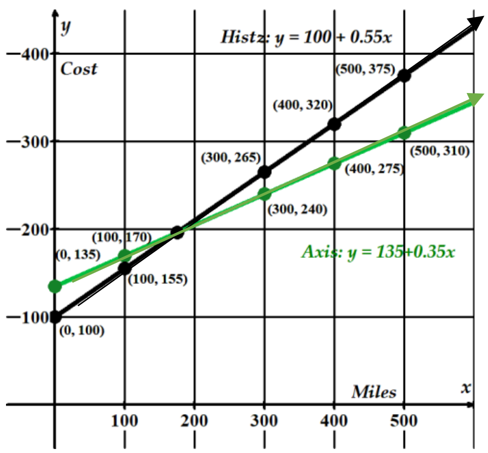
$y = 100 + 0.55x$

b)

Miles x	Axis Cost : $y = 135 + 0.35x$	Ordered pair (x, y)
0	135	(0, 135)
100	170	(100, 170)
200	205	(200, 205)
300	240	(300, 240)
400	275	(400, 275)
500	310	(500, 310)

Miles x	Histz Cost : $y = 100 + 0.55x$	Ordered pair (x, y)
0	100	(0, 100)
100	155	(100, 155)
200	210	(200, 210)
300	265	(300, 265)
400	320	(400, 320)
500	375	(500, 375)

c)



d)

$135 + 0.35x = 100 + 0.55x$

$35 = 0.20x$

$175 = x$

At 175 miles, the cost would be the same for both rental companies

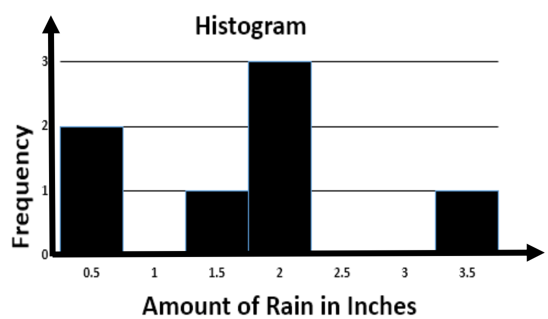
5) **a)** Days of the week

b) Amount of rain in inches

c)

Quantitative variable: Amount of rain in inches	Tally	Frequency (f)
0.5	//	2
1		0
1.5	/	1
2	///	3
2.5		0
3		0
3.5	/	1

$\sum f = 7$

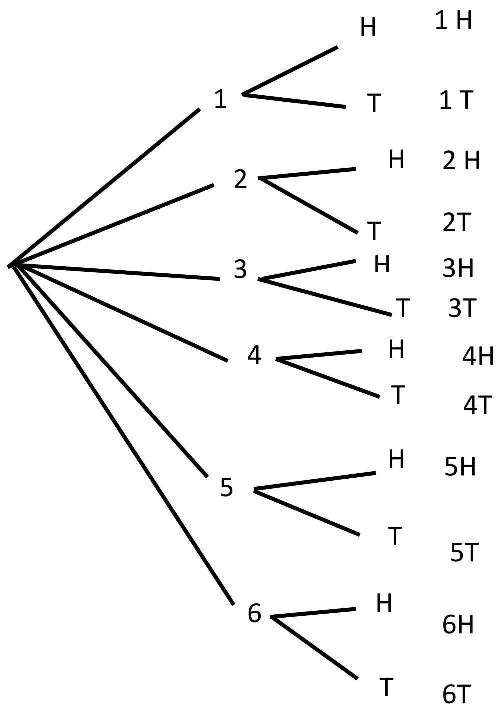


d) The mean is $\bar{x} = \frac{\sum x}{n} = \frac{2(0.5)+1(1.5)+3(2)+1(3.5)}{7} = 1.71$. The average amount of rain is 1.71 inches

To find the median, we have to order from least to greatest, 0.5, 0.5, 1.5, 2, 2, 2, 3.5. Since there are 7 days in a week, the median is the middle data entry $\frac{7}{2} = 3.5 = 4\text{th}$ place. .

The 4th number of the list is 2. So the median is 2 inches of rain.
The mode is the amount of rain that occurs the most which is 2 inches.

6) a)



b) List the sample space. {1H, 1T, 2H, 2T, 3H, 3T, 4H, 4T, 5H, 5T, 6H, 6T}

c) 6 (2) = 12 different outfits.

7) (26)(26)(26)(10)(10)(10) = 17,576,000 different passwords can be created.

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